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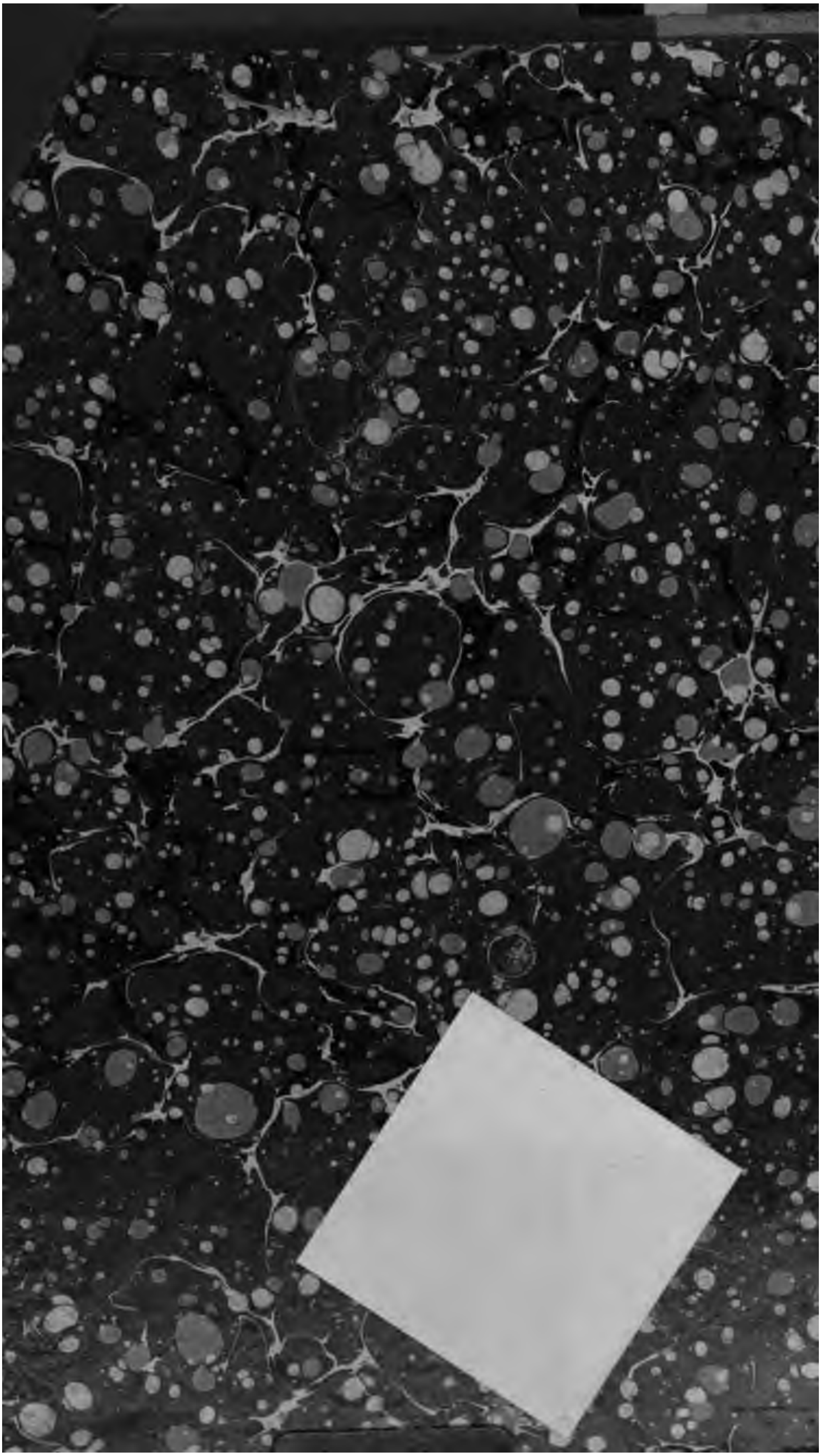

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BULLETIN OF THE
NEW YORK
MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE.

EDITED BY
THOMAS S. FISKE, ALEXANDER ZIWET.

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BULLETIN OF THE

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INAUGURAL ADDRESS

DELIVERED AT THE GENERAL SESSION OF THE CONGRESS OF MATHEMATICS AND ASTRONOMY AT CHICAGO, AUGUST 21, 1893.

BY PROF. FELIX KLEIN.[*]

THE German Government has commissioned me to communicate to this Congress the assurances of its good-will, and to participate in your transactions. In this official capacity allow me to repeat here the invitation given already in the general session to visit at some convenient time the German university exhibit in the Liberal Arts Building.

I have also the honor to lay before you a considerable number of mathematical papers, which give collectively a fairly complete account of contemporaneous mathematical activity in Germany. Reserving for the mathematical section a detailed summary of these papers, I mention here only certain points of more general interest.

When we contemplate the development of mathematics in this nineteenth century, we find something similar to what has taken place in other sciences. The famous investigators of the preceding period, Lagrange, Laplace, Gauss, were each great enough to embrace all branches of mathematics and its applications. In particular, astronomy and mathematics were in their time regarded as inseparable.

With the succeeding generation, however, the tendency to specialization manifested itself. Not unworthy are the names of its early representatives: Abel, Jacobi, Galois, and the great geometers from Poncelet on, and not inconsiderable are their individual achievements. But along with its rapidly growing development the science departed more and

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[*] First published in *The Monist*, Vol. IV. pp. 1–4.

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more from its original scope and purpose, threatening to sacrifice its earlier unity and to split into diverse branches. At the same time the attention bestowed upon it by the general scientific public diminished in equal proportions. It became almost the custom to regard modern mathematical speculation as something having no general interest or importance; and the proposal was even made that, at least for purposes of instruction, all results be formulated from the same standpoints as in the earlier period. Such conditions were unquestionably to be regretted.

This is a picture of the past. I wish on the present occasion to state and to emphasize the fact that in the last two decades a marked improvement from within has asserted itself in our science, with constantly increasing success.

The matter has been found simpler than was at first believed. It appears, indeed, that the different branches of mathematics have actually developed not in diverging but in parallel directions; that it is possible to combine their results into certain general conceptions. Such a conception is that of the *function*—in particular that of the analytical function of the complex variable. Another conception of perhaps the same range is that of the *group*, which just now stands in the foreground of mathematical progress. Proceeding from the idea of groups, we learn more and more to co-ordinate and connect different mathematical sciences. Thus, for example, geometry and the theory of numbers, which long seemed to represent antagonistic tendencies, no longer form an antithesis, but have come in many ways to appear as different aspects of one and the same theory.

This unifying tendency, originally purely theoretical, must inevitably extend to the applications of mathematics in other sciences, and on the other hand is sustained and reinforced in the development and extension of these latter. I presume that specific examples of this interchange of influence between pure and applied mathematics may be not without interest for the members of this general session, and on this account have selected for brief preliminary mention two of the papers which I have later to present to the mathematical section.

The first of these papers (by Dr. Schönflies) presents a review of the progress of mathematical crystallography. Sohncke, about 1877, treated crystals as aggregates of congruent molecules of any shape whatever, regularly arranged in space. In 1884 Fedorow made further progress by admitting the hypothesis that the molecules might be in part inversely instead of directly congruent. In the light of our modern mathematical developments this problem is one of the theory of groups, and we have thus a convenient starting-point for the solution of the entire question. It is simply necessary to enumerate all discontinuous groups contained in the so-called principal group of space-transformations. From this point of view Dr. Schönflies has treated the subject in a text-book (1891), while in the present paper he discusses the details of the historical development.

In the second place, I will mention a paper which has more immediate interest for astronomers, namely, a *résumé* by Dr. Burkhardt of the relations between astronomical problems and the theory of linear differential equations. This paper deals with those new methods of computing perturbations that were brought out first in your country by Newcomb and Hill; in Europe, by Gyldén and others. Here the mathematician can be of use to the astronomer, since he is familiar with linear differential equations and is trained in the deduction of strict proofs; on the other hand, the professional mathematician finds here much to be learned. Hill's researches involve indeed—a fact not yet sufficiently recognized—a distinct advance upon the current theory of linear differential equations. To be more precise, the interest centers in the representation of the integrals of a differential equation in the vicinity of an "essentially" singular point. Hill furnishes a practical solution of this problem by the aid of an instrument new to mathematical analysis—the admissibility of which is, however, confirmed by subsequent writers—the infinitely extended, but still convergent, determinant.

Speaking, as I do, under the influence of our Göttingen traditions, and dominated somewhat, perhaps, by the great name of Gauss, I may be pardoned if I characterize the tendency that has been outlined in these remarks as a *return to the general Gaussian programme.* A distinction between the present and the earlier period lies evidently in this: that what was formerly begun by a single master-mind we must now seek to accomplish by united efforts and coöperation. A movement in this direction was started in France some time since by the powerful influence of Poincaré. For similar purposes we founded in Germany a mathematical society, three years ago, and I greet the young society in New York and its Bulletin as being in harmony with our aspirations. But mathematicians must go still farther. They must form international unions, and I trust that this present congress at Chicago will be a step in that direction.

FELIX KLEIN.

INSTRUCTION IN MATHEMATICS IN THE UNITED STATES.

BY PROF. T. H. SAFFORD.

WITHIN a few years the general standard of mathematical teaching in this country has greatly risen. The history, in brief, of this subject can be easily narrated. Before the Revolution, our colleges were colonial English, and were somewhat like the smaller colleges of Oxford and Cambridge. The attempt was made to teach Newton's philosophy, with its introductory mathematics, including Euclid. These efforts were only partially successful, because of the low standard of scholarship in the colleges. Later, the country became independent; and we began to borrow ideas from our friends the French. This culminated, still later, in the mathematical text-books of Farrar, Davies, Loomis, and others; essentially based on French models, with that lack of demonstrative rigor which was permitted early in the century. The American mathematicians Bowditch, Peirce, and Chauvenet, with their disciples and followers, introduced the writings of Gauss and the Germans and other modern Continental writers, including of course Cauchy, Leverrier and other great French geometers, to this country; and the Harvard, Yale, and Baltimore schools of mathematics, together with many smaller colleges, have done much to make the study of higher subjects not only possible but necessary, at least on the part of our future professors and teachers of the subject.

The object of the present article is to give some practical hints, with especial reference to the existing state of things. In the first place, our colleges nominally require much which is practically ignored by many teachers of preparatory schools. A student is often admitted to college with many "conditions" in the mathematics. This simply means that he is ill prepared in the subject; and is required, before he can be fully matriculated, to pass a rather superficial re-examination. The teaching in the schools is based very much on the college examination papers; and hence it does not meet the ever-growing requirements of the college course. Again, it is tolerably easy to get a pupil past an entrance-examination, if he is allowed to forget as soon as may be all that is then required. The standard of teaching which entrance-examinations tend to produce is, therefore, a superficial one; and the rise of standard more nominal than real. Certain papers in algebra which the writer has seen are admirably calculated to dampen the enthusiasm of any pupil who attempts to do them ; and at best to produce a kind of mathematical teaching which is adapted to the school and not at all to life.

Again, the subjects taught in schools are not rightly arranged. The "higher arithmetic" is quite useless; it is a disguised algebra, and one which is tied down to the American dollar as its main material. The moral effect of this, so far as it has any, is bad ; and the results, for those who are to go into business, either bad or none at all.

Algebra, philosophically considered, is a higher arithmetic; and the pupil who is thoroughly taught algebra is quite competent to solve all business riddles much better than he can by the present rather senseless method. Here, as well as anywhere, the remark is in place, that the teachers of mathematics in this country, high and low, are too often persons who do not understand pedagogy, do not pay much attention to the mutual adaptation of the subject and the mind, and attempt to crowd utterly indigestible stuff into the rather limited intellect of the average pupil. The psychologists have long pointed out the fact that technical mathematics is not a good study for training, in so far as it is too abstract ; geometry, in which the things to be learned are sensibly represented, is far better than arithmetic, which is very abstract ; Euclid's geometry is rightly arranged, in that the seventh, eighth, ninth, and tenth books contain material which is dealt with superficially in our ordinary arithmetics. The neglect of geometry in our schools is a very serious defect. In order to teach the subject as well as algebra is taught, much more time must be given than at present is allowed. The early portions, the rudiments, or *Formenlehre* as the Germans call it, must be taken up in advance of the logical portion ; and in fact along with quite rudimentary arithmetic. And the easy early books of demonstrative geometry should come along with the early parts of algebra. The ordinary practice of deferring all geometry till algebra has been far advanced is altogether bad ; it serves only the convenience of schoolteachers and by no means the development of the pupils' minds. Even ordinary reckoning is ill-taught. Mental arithmetic is subordinated to written ; while it is an essential constituent of written, and should have priority over it in every respect. The written arithmetic is the servant and not the master.

Many of these things would be bettered if our teachers were more rigidly examined as to their preparation. One can pretty surely infer that, as a rule, they are instructing in classes too far advanced for their knowledge. Those who now teach in the so-called high schools should teach there no longer, unless they have had an education which has fitted them to completely master what they are teaching. A text-book knowledge, or a mere school knowledge of the subjects, is not enough; they should have had experience in their practical every-day application. For instance, a good teacher of logarithms and trigonometry is necessarily one who is expert with logarithms and has employed trigonometry in some of its engineering or physical applications. The best way to train an able young man or woman in these subjects up to the point required is to give him or her observatory practice up to, for instance, the calculation of a correct orbit. Then the plane trigonometry will not be an abstract and meaningless matter. The same end may, very likely, be attained by the use of a physical laboratory or in a school of surveying which practically admits that the earth is round, and that the plumb-lines at different stations are not parallel.

The higher mathematics are important, as college electives and university studies, for all who are to be college or normal school professors; and the future high school teacher will do well to learn at least the rudiments of the calculus.

But the pure mathematicians are a little apt to overestimate the importance of their subject as a means of education. Mathematical training is best given by rather elementary exercises, at least to most students; one principle at a time can be thus taught, and the mind will thus receive, appropriate, and apply it with more spontaneity. The heuristic method, that in which the pupil discovers a principle for himself, is far more effective than the dogmatic; and far more than that according to which large masses of mathematical formulæ are committed to memory. There are still teachers actively at work who require pupils to commit to memory a great many geometrical propositions, with corollaries and scholia, and even the numbers attached to the same in the text books; a method which has wrought vast mischief in this country. Two subjects in which the present writer is quite certain that the heuristic method is applicable are mental arithmetic (according to Grube) and the modern geometry. The integral calculus would be included but for the difficulty (at present) which arises from the long training by another method to which all our pupils are subjected in algebra.

The great colleges have "raised the standard" in this subject rather too rapidly; so that the students come up "crammed" with it rather than trained in it. It is now quite time to insist upon a different method in preparatory schools; and to do this by adding oral examinations to those now given by printed papers. In that way it is probable the requisite thoroughness in the elements can, little by little, be attained; especially if college professors will put themselves on the pedagogic standpoint. The weakness of mere written examinations is as much a didactic commonplace as that of mere oral ones; witness the universal method of examining for the higher degrees. The college electives in mathematics ought to include subjects of moderate difficulty as well as advanced ones. Among the most valuable, which are frequently neglected, are conic sections synthetically treated, and kinematics as a branch of pure geometry. Whether the applications of the calculus contained in the ordinary text-books are sufficient is a little questionable; Messrs. Rice and Johnson have done a good work for the Naval school in preparing examples adapted to their pupil's daily work; and it is much to be wished that the other writers of our text-books in this subject would exhibit equal originality. American text-books of much value in other respects are far too largely copied from the latest English ones; and it seems a pity that so thoroughly artificial an institution as the Cambridge mathematical tripos should become as it were a model for our teachers and examiners.

In general the best principle to adopt in mathematical training of any grade is that the matter taught shall have as definite a reference as possible to other studies or pursuits of the pupil. There are few who can derive much benefit from the higher mathematics unless they are either interested in the subject for itself or for its applications. It is quite certain that those who are not interested in mathematics do not derive much benefit from its study. Any one can be taught geometry by a competent instructor; elementary algebra, although more difficult, has so many applications in practical life that it can be made interesting: plane trigonometry, too, is brought nearer educated men by its applications. Analytical geometry is less interesting to those who are not technically mathematicians or students of physics; and usually cannot be carried much beyond conic sections: these curves can be best taught to average college students by the synthetic method, by which they are rendered more tangible and interesting. All subjects beyond this should be elective. The "modern mathematics" are in the main useful only to those who are preparing themselves to be teachers of the science or of physics or astronomy; but for these they are indispensable.

We may then sum up what has been said in a few words. Mathematical preparation for college must be more thorough. The courses for this purpose must be better organized. The method must call upon the pupil to originate when possible; and must not encourage cramming. The teachers must be better prepared for their work, especially in mastery of the subject and its applications. College professors must not require the average students to do technical work. They must remember that if the power of abstraction fails their pupils they are often tempted to superficial and insincere study. They must also be willing to teach the higher subjects to small classes only.

Practical applicability in physics and astronomy must be the test by which it is decided what can be demanded of a large majority of their scholars; for it is these sciences which render the abstractions of pure mathematics not only intelligible but interesting to many who have not the utter disregard of the outside world which is characteristic of the pure mathematician.

ON THE TEACHING OF ELEMENTARY GEOMETRY.

Plane Geometry. On the heuristic plan. By G. I. HOPKINS. Boston, D. C. Heath, 1891.

Elementary Synthetic Geometry. By N. F. DUPUIS. New York, Macmillan, 1889.

Introductory Modern Geometry. By W. B. SMITH. New York, Macmillan, 1893.

Elementary Synthetic Geometry. By G. B. HALSTED. New York, Wiley & Sons, 1892.

NOWHERE has the conflict between the forces of conservatism and radicalism waged hotter than in the domain of geometry. The nature of the axioms, the character of the reasoning employed, the method in which the science shall be taught, have each given occasion for many a battle. Peace is not yet, but progress toward it is discernible.

To begin with, it is coming to be generally admitted that geometry is a physical science and that the truth of certain of its axioms, instead of being necessary and self-evident, is dependent upon the nature of the space and our means of observation. Space being an hypothesis that the mind makes to explain phenomena, the character of space depends upon the character of the phenomena observed. That the phenomena that give our space-conceptions should be observed, and carefully too, the struggle for existence has inexorably compelled.

Then, too, it is here and there perceived that the reasoning of geometry, of which the characteristic is to spin out as many conclusions from as few data as possible, is not ideal. If observation, it is said, without our scarcely being aware of it, has given us our data, may it not equally have been playing a part in all our reasoning? Do we not reason rightly because we perceive rightly, so that geometry, as Gauss would have it, is "the science of the eye"? Would it not be more logical to consciously and avowedly use our eyes? And is it not safer to observe much and draw few conclusions, rather than little and draw many?

As for the manner of teaching, the number of those who would be content to set a student to memorize the demonstrations of a text grows daily less. That a student should observe and compare, and then draw his own conclusions; that he should have continual opportunity to apply his knowledge; that he should test both his own guesses and the statements of the book by careful constructions, and use these same constructions to suggest new theorems and methods,—all this bids fair to become a matter of course.

Of this general progress and of attempts at improvement in many minor details, the books whose titles head this article furnish instances. All deem it necessary to state that figures can be moved about in space without changing their size or shape. Mr. Dupuis and Mr. Halsted each distinctly calls this an assumption. Mr. Smith goes further and intimates that space may be boundless without being infinite. Moreover, he states some of the properties of space of uniform positive and of uniform negative curvature. All give plenty of problems for the student to work, and Mr. Hopkins's book is indeed mainly a collection of problems. All except Mr. Hopkins give some prominence to modern synthetic geometry, while Mr. Halsted gives the student a taste of even the more recent Lemoine-Brocard geometry. Let us take up these books in some detail.

Though Mr. Hopkins does not go beyond the time-honored bounds of elementary geometry, he claims a substantial improvement by presenting the subject *heuristically*. It surprises one, then, to find the book beginning with ten or a dozen pages of definitions and axioms. Does not heuristic treatment require that technicalities should be brought in by degrees rather than all at once? Again, we find given, for the student to demonstrate, at the very start, such propositions as, "all right angles are equal;" "if two angles are equal, the complements of those angles are equal," propositions whose truth to the student will seem as plain as any demonstration that can be given. Surely it would be better at first to confine the student to reasoning from what seemed self-evident to what did not. He would at least not run so great danger of thinking that in reasoning the chief essential was formality. As a further example of the heuristic method, the author expresses his "firm belief that mathematicians have no right to amalgamate the *proportion form* and the *equation form* of expression." Yet we are not told why; on the contrary, the two are said to be equivalent! Quite as remarkable is the author's original demonstration of the Pythagorean proposition, which he makes depend upon this:

If, in any circle, there be drawn a diameter perpendicular to a chord, and if from one end of that diameter a second chord be drawn intersecting the first, then the rectangle on this second chord and that segment of it that meets the diameter is equivalent to the square on the concurring segment of the diameter increased by the square of half the first chord!

What is there heuristic about the book? Well, perhaps that certain of the demonstrations are given by means of leading questions; or that definition and consideration of limits and symmetry are relegated to the appendix; or that, now and then, there is such excellent advice to the student as, "use the most unfavorable figure;" but chiefly, I think, that the book is a collection of problems, that leaving out these it is impossible to find a continuous text for memorizing. If, by this, a few more teachers are driven to requiring problem working of their students, the book will do good.

Mr. Dupuis's geometry has been prepared with extreme care and covers, with admirable thoroughness, much ground. Some will doubtless object to his treating distance and direction as simple conceptions; but simpler they certainly are than the reasoning that proves them mysterious, reasoning which, after all, winds up by adopting for Euclid's space precisely the ordinary common-sense conceptions. The state of the matter is this. Certain notions are derived from our race-experience, among them distance and direction. These notions profound investigations have shown to be compatible with only "the dreary infinitudes of homoloidal space." But what have we to do with any other space in elementary geometry? Ought not a student, must he not, in fact, really begin with his own race- and experience-given notions? When he has learned to reason from these as a basis it will be time to think of how to soar above such petty restrictions into the heaven of the nth dimension. Once break the bounds and where shall we stop? Is there, after all, any more warrant for assuming that space is alike throughout, *homœoidal*, than for saying that it is homoloidal? Can we even maintain that it possesses the property of elementary flatness? These are matters for the professional mathematician, and it is not necessary to confuse the learner with them.

The usual tedious calculation of π is omitted, the student being referred elsewhere for this. Algebra is skillfully worked in with the geometry, and there is a good chapter on the interpretation of algebraic forms. The allusions to mathematical instruments are valuable, if only the instruments themselves can be put into the student's hands for actual practice. The early introduction of *sine* and *cosine* might well be followed up by a short course in plane trigonometry before going on with the rest of the book. The concluding section on "geometric extension" (modern geometry) is perhaps as good as can be, if with the circle the other conics are not to be treated. Taken all together the book very well serves the author's stated purpose of an introduction to the modern works on analytical geometry.

Mr. Smith's book has already been reviewed in the BULLETIN.

In reading Mr. Halsted's book it is difficult to rid one's self of the impression that the author somewhat scorns conservatism.

Even the language is original. An indefinite straight line is a *straight*, a limited portion of it is a *sect*, and points upon it are *costraight*. Central symmetry is *symcentry* and a symcentral spherical quadrilateral is a *cenquad*. One is reminded of a certain colloquy beginning: "Do you abbrev.?" "Cert."

The demonstration of all the usual cases of the congruence of triangles is given in eight lines of text with no reference to a diagram. There are, however, on the same page several figures evidently intended for illustration. Demonstration in general terms is, indeed, a marked feature, and a good one, of the book. Of course the student should be exercised in applying these to diagrams, while he should also have practice in changing back from forms with diagrams to forms in general terms.

The arrangement of the subject matter illustrates the fact that in geometry, as in other sciences, one can begin almost anywhere and go in almost any direction, if he will but proceed circumspectly. And if putting pure spherics near the beginning only leads some to realize that solid geometry can and ought, to some extent, to be taught along with plane, the way will be paved for a distinct advance over the usual presentation.

The chief advantage in bringing in chapters on modern and recent geometry is to teach the progressiveness of the science. The end would have been further secured by historical notes, and the student would have the further advantage of learning how modern and how recent the theorems were.

In books having so much that is unusual in matter and arrangement as those of Dupuis, Smith and Halsted, an index is especially helpful, and it is gratifying to find that each contains one. All would be improved by having, in addition, synopses and syllabi ; while more abundant references to first sources would be valuable to teachers and advanced students.

The book that shall present geometry heuristically is, I fear, an unattainable ideal. Each student would have to be specially written for, and only his particular teacher, daily carefully watching his mental growth, could do it. Even this teacher would run a continual risk of interfering with the student's development. Suppose the student goes wrong at times. An occasional mistake is not necessarily harmful ; further investigation is sure, at last, to correct it. Only let the students be kept interested and a year's work will teach them much.

But the teacher must study his class. No amount of routine will quite kill their originality. Howsoever careful and complete the memorizing, the precedence of ideas in the students' minds will not be just that in the book, and each student will actually, in so far as he does learn anything, be learning in his own way. When even teachers and bookmakers differ so much, what must we look for in the yet untrained boy.

Take a case in point.

During the past year I had a small class in geometry. The repugnance of many of them to demonstrating what seemed quite plain without it interested me, and I asked what propositions in their geometry (Byerly's Chauvenet) seemed to them to need no demonstration. Here are the answers on the first book. They had studied three books.

Out of 13 students, 13 thought I self-evident; 13, II; 12, III; 7, IV; 9, V; 7, VI; 7, VII; 10, VIII; 11, IX; 6, X; 6, XI; 8, XII; 6, XIII; 4, XIV; 5, XV; 9, XVI; 12, XVII; 9, XVIII; 6, XIX; 4, XX; 8, XXI; 12, XXII; 11, XXIII; 3, XXIV; 4, XXV; 3, XXVI; 1, XXVII; 5, XXVIII; 9, XXIX; 8, XXX; 6, XXXI; 5, XXXII.

There was not a single proposition that some one did not think self-evident, and in the entire five books of plane geometry there were only six, viz.: X of book III; VIII, IX, and X of book IV; IV and IX of book V.

Evidently the list of axioms could be considerably enlarged with perfect satisfaction to the average student. The advisability is shown, it seems to me, of pointing out the reasons for undertaking certain demonstrations.

To still further test the class I drew up the following questions:

1. One of two iron balls is twice as far through as the other. How many times as much does it weigh?

2. How many times as much leather does it take to cover it?

3. If one angle of a triangle is twice as large as another, is the side opposite the first angle twice that opposite the second?

4. Two sides of one triangle are respectively equal to two sides of another triangle, but the third side of the one is once and a half the third side of the other. Is the angle opposite the third side in the first triangle once and a half that opposite the third side in the second triangle?

5. If one chord of a circle is twice as large as another, will it be twice as near the centre?

6. Will the two lines that trisect the vertical angle of an isosceles triangle also trisect the base?

7. Which will stand the most firmly on an uneven floor, a three-legged, a four-legged, or a five-legged table?

8. Which will take the more stone to build a wall 3 feet high, 1 foot thick, and a mile long, on a level or over a hill? Distance is measured along the wall.

9. If you wanted to go the nearest way to a point due east, would you go directly east?

10. Can you cut out of a potato a flat-faced solid with 5 edges? with 6? with 7? with 8?

11. Can you cut one out all of whose faces are six-sided?

12. How many braces does it take to stiffen a three-sided plane figure? a four-sided? a five-sided?

13. Are lines in space perpendicular to the same line parallel?

14. One of two square pyramids is twice as high as the

other, but its base is half as far across. Does it take more or less stone to build it?

In answering these questions, students were asked to give their first impressions, their second impressions, any reasons, even vague ones, that they might have for their answers.

Replies were received from twenty. I give a part of the results:

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Number giving right answer	4	7	6	2	8	3	13	1	6	2	4	0	3	5
Number giving right reason	2	2	5	0	3	1	0	1	1	2	0	0	0	3

The largest number correctly answered by any one student was 8, and for all but one of these answers his reasons were correct. On first impressions he would have answered but two rightly and have been doubtful about one of these.

The tendency to accept as true statements resembling true ones is clearly marked. This is an exceedingly useful tendency; by its aid we frame inductions. But there needs to go along with it the habit of taking the resembling statements only provisionally, to be carefully tested at the first opportunity. In geometry the supreme test is, of course, "Can a valid reason be given?" But, while awaiting that, there is readily available the test of careful drawing and observation. Of this there cannot be too much. But with this there should go reasoning from figures that are purposely drawn false, that are obtrusively, what all figures are when innerly viewed mere diagrams to symbolize the relations under consideration.

I have referred to the importance of connecting geometry with its practical applications. Here it seems to me that all our texts, even these recent ones, are weak. The early introduction of trigonometry and mensuration is only one of many opportunities offered. The triangle of geometry stands for the triangle of forces, the theory of proportional lines leads at once to graphical methods of computation, that of intersecting lines and planes leads to descriptive geometry and perspective, while these, in turn, furnish the best possible introduction to modern projective geometry. Such are among the many vistas into the surrounding universe that are to be gained from the highway of geometry. Why should these be shut out by the close walls of prejudice and custom on either hand? There is not time for these outlooks! Why, the way to save time is to make the teaching interesting. To this end, varied applications and illustrations are the most effective means.

ELLERY W. DAVIS.

THE MATHEMATICAL CONGRESS AT CHICAGO.

THE mathematical section of the Congress on Mathematics and Astronomy held in Chicago from August 21st to 26th was of the highest interest to all present, particularly on account of the active participation of Prof. Klein, of Göttingen. Only a brief outline of the proceedings can be presented here, but it is hoped that a full official report of the proceedings will ultimately be published.

Monday's session was devoted to preliminary addresses and to organization, Prof. Klein referring in his introductory address* to two of the special papers presented :

Gruppentheorie und Krystallographie, by Prof. Schönflies of Göttingen, and

Ueber einige mathematische Resultate neuerer astronomischer Untersuchungen, insbesondere über irreguläre Integrale linearer Differentialgleichungen, by Dr. Burkhardt of Göttingen.

The so-called "structure theory" of crystals deals essentially with a problem of the theory of groups, namely, with the enumeration of all discontinuous sub-groups which may be formed from the group of space-movements, combined also with reflection. Dr. Schönflies has already (1891) made a comprehensive presentation of the theory from this standpoint.

Dr. Burkhardt gives a critical account from a mathematical standpoint of the interesting advances made in the last twenty years by Hill, Glydén, and others in the field of perturbation calculation. These authors having employed linear differential equations for defining their trancendental functions, their investigations stand in close relations with developments in the theory of these equations made in more recent years from the purely mathematical side.

At the second session (August 22) the following papers were presented :

Ueber die Theorie der algebraischen Invarianten, by Prof. Hilbert of Königsberg, giving an account of the results reached in the investigations on the finiteness of the form systems, in which by the introduction of fundamentally new methods he has passed far beyond the previous standpoint.

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* See this number of the BULLETIN, p. 1.

Zur Theorie der ganzzahligen algebraischen Gleichungen, by Prof. Weber, of Göttingen, giving an elementary proof that there is an infinite number of equations of prime order and with integral coefficients, which in the domain of rationality have no "Affect,"—equations, that is, whose Galois group is the symmetrical group.

Ueber die arithmetisch-algebraischen Tendenzen Leopold Kroneckers, by Prof. Netto, of Giessen, illustrating the tendencies followed by Kronecker, particularly in his later years, to reduce the entire field of mathematics to the *relations between integral numbers*, and to consider only such mathematical operations legitimate as involve but a finite number of steps.

Ueber die Reduction der binären quadratischen Formen, by Prof. Hurwitz of Zürich, a derivation, by a particularly simple and elegant geometrical method of known theorems on the reduction of the binary quadratic forms.

On Fifth-power Numbers, whose Sum is a Fifth Power, by Dr. Artemas Martin of the U. S. Coast and Geodetic Survey, giving special numerical cases.

On the Algebraic Solution of Equations, by Prof. Sawin of Evansville, Wisconsin.

Aeltere und neuere Untersuchungen über Systeme complexer Zahlen, by Prof. Study of Marburg. The author deals with those complex numbers for which the product of two units is a linear combination of the original units ; he gives the historical development of the theory, inviting special attention to the relations of the considerations involved to Lie's theory of continuous transformation-groups.

On the Definitions of the Trigonometric Functions, by Prof. Macfarlane, of the University of Texas.

The session of Wednesday opened with a paper on *Modern Graphical Developments*, by President Eddy of the Rose Polytechnic Institute, reviewing the development of graphical methods since the time of Monge, with particular reference to M. Lévy's "La statique graphique et ses applications aux constructions."

Some Salient Points in the History of Non-Euclidean and Hyper-Spaces were presented by Prof. Halsted of the University of Texas, with an account of the work of an Italian priest, Saccheri, published in 1733, containing a "statement of propositions in Lobatscheffsky's Non-Euclidean Geometry, with their synthetic proof in pure geometric style." *

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* See BULLETIN, vol. II., p. 44.—Editor.

Prof. Study of Marburg then presented a paper on *Some Researches in Spherical Trigonometry*, dealing with the formulæ of the latter from the standpoints of the modern theory of functions and of the theory of groups and exhibiting close relations to various other branches of mathematics: the theory of orthogonal substitutions, that of desmic surfaces of the fourth order, that of the theta-relations, etc. (See the author's detailed account in the Abhandlungen der sächsischen Akademie, 1893.) This was followed by a paper.

On Interpolation Formulæ and their Relation to Infinite Series, by Professor Echols of the University of Virginia; by a

Résumé de quelques résultats relatifs à la théorie des systèmes récurrentes de fonctions, by Prof. Pincherle of Bologna.

Sur une intégrale définie qui représente la fonction $\zeta(s)$ *de Riemann*, by Prof. Lerch of Prague.

Ueber Eigenschaften von ganzen Zahlen, die durch räumliche Anschauung erschlossen sind, by Dr. Minkowski of Bonn, giving a preliminary account of the present state of the geometrical investigations on parallel grates (Parallelgitter) in space of n dimensions, by means of which he has derived a series of remarkable results in the theory of numbers.

Ueber die nothwendigen und hinreichenden Bedingungen für die Entwickelbarkeit von Functionen einer reellen Variablen nach der Taylorschen Reihe, by Prof. Pringsheim of Munich. The author gives first a review of our present knowledge of criterions of convergence of infinite series, then derives the conditions named in the title.

Consecutive und coincidirende Elemente einer algebraischen Curve, by Prof. Nöther of Erlangen, explaining anew the methods he has employed in the investigation of singular points, and showing the simplicity with which the chief theorems may be expressed.

Thursday's programme included the following papers:

The Principles of the Elliptic and Hyperbolic Analysis, by Prof. Macfarlane of the University of Texas, in continuation of his earlier paper.

On Weierstrass's Systems of Abelian Integrals of the First and Second Kinds, by Prof. Bolza of the University of Chicago, a precise derivation of certain fundamental theorems, using the methods of Weierstrass.

Fortschritte in der Theorie der linearen Differentialgleichungen, by Dr. Heffter of Giessen, a review of the researches of Fuchs and other German writers during the past five years.

Automorphe Functionen und Zahlentheorie, by Dr. Fricke of Göttingen, giving an account of the relations to arithmetical developments of the numerous discontinuous groups occurring in the theory of automorphic functions, so far as such relations have been investigated by himself and others.

Zur Transformation fünften Grades der hyperelliptischen Functionen erster Ordnung, by Prof. Krause of Dresden.

Sur quelques propositions fondamentales de la théorie des fonctions elliptiques, by Prof. Hermite of Paris, giving a new derivation of the addition theorem of the elliptic functions for any assigned biquadratic form under the radical sign in the elliptic integral.

The following papers were included in the programme for Friday:

On Orthogonal Substitutions, by Prof. Taber of Clark University, giving the determination of all possible real orthogonal substitutions (and of imaginary orthogonal substitutions with two, three, four or six variables) rationally in terms of the minimum number of parameters; the determination of all symmetric orthogonal substitutions of n variables rationally in terms of $\frac{1}{2}n\,(n-1)$ parameters; and certain generalizations of Stieltjes's theorem, with an exponential representation of orthogonal substitutions.

On a Quaternary Group of 2520 Linear Substitutions, by Prof. Maschke of the University of Chicago, giving an account of the complete form system of that quaternary group of 2520 substitutions occurring in line geometry.

On Simple Groups, by Prof. Cole of the University of Michigan, describing a new simple group of 504 substitutions of nine letters.

A Doubly-infinite System of Simple Groups, by Prof. Moore of the University of Chicago. The theory of the elliptic modular functions gives a simple group (known since the time of Galois) of $\frac{q(q^2-1)}{2}$ substitutions, where q is any prime number greater than 2. By introducing substitutions with Galois imaginaries, the author obtains an entire series of new groups with simple $\frac{q^n(q^{2n}-1)}{2}$ substitutions. These results hold also for $q=2$ if the denominator 2 be omitted. For $q=2$, $n=3$, we obtain the group discussed by Prof. Cole.

At the final session (August 26) the following papers were presented:

A Formulary for an Introduction to Elliptic Functions, by Prof. Stringham of the University of California. The author develops a system of formulæ for the reduction of the elliptic integral of the first kind to different normal forms with corresponding functional notations, distinguishing the different types by the particular Cayleyan transformations employed in deriving them. The earlier formulæ of Jacobi and Abel are then derived as well as those of Weierstrass.

A Construction of Galois's Group of 660 *Elements*, by Mr. Joseph de Perott of Clark University.

Tabellen von endlichen continuirlichen Transformationsgruppen, by Prof. Meyer of Clausthal. One of the most remarkable results of the theory of Lie is that for a given number of variables, for example, in the plane, there is only a limited number of distinct types. While Lie, however, has developed these equations only in the abridged infinitesimal form, the present paper contains complete lists of these types in finite form.

Einige Sätze vom Schwerpunkt, and *Der pythagoräische Lehrsatz in mehrdimensionalen Räumen*, by Prof. Schlegel of Hagen.

Considérations générales sur la mesure de la simplicité dans les sciences mathématiques et applications à l'évaluation théorique de la simplicité des tracés géométriques, and *Règle des analogies dans le triangle et transformation continue*, by M. Émile Lemoine of Paris.

Sur l'équation des lignes géodesiques, by Prof. Weyr of Prague.

Nomographie: Sur les équations représentables par trois systèmes rectilignes de points isoplèthes, by M. d'Ocagne of Paris.

Note concerning Arithmetical Operations involving Large Numbers, by Rev. T. M. Pervouchine of Kasan.

Quelques formules relatives aux operations de polaire, by Prof. Capelli of Naples.

Sul moto di rotazione di un corpo rigido attorno ad un punto fisso, by Prof. Paladini of Pisa.

Concerning the Development of the Theory of Groups during the last Twenty Years, by Prof. Klein of Göttingen. From his Erlangen Programme of 1872 (recently translated in the BULLETIN) as a starting-point, Prof. Klein discusses briefly the development of the theories both of continuous and of discontinuous groups, referring to his recent lecture course, Höhere Geometrie II (soon to be lithographed).

After a unanimous and enthusiastic vote of thanks to Prof. Klein for his efforts in promoting the interest and success of these meetings and for the courtesies extended by him as Prussian commissioner at the Exposition, and after congratulatory remarks by Prof. Story, the section adjourned.

On three afternoons during the sessions of the Congress, Prof. Klein gave highly interesting expositions of the very complete Exhibit of the German Universities at the Liberal

Arts Building. Two afternoons were devoted mainly to the explanation and discussion of mathematical models and other appliances, of which an extensive collection had been arranged by Profs. Klein and Dyck. Many of the models were unfamiliar to those present, and the opportunity for their examination was highly appreciated.

The officers of the mathematical section were:

Honorary President, Prof. Klein of Göttingen.

President, Prof. Story of Clark University.

Vice-President, Prof. Moore of the University of Chicago.

Secretary, Prof. Tyler of the Massachusetts Institute of Technology.

Executive Committee, the officers and Prof. White of Northwestern University.

Besides the officers just named the following were among those attending the congress: Profs. Study of Marburg, Halsted and Macfarlane of the University of Texas, Eddy of the Rose Polytechnic Institute, Bolza and Maschke of the University of Chicago, Paladini of Milan, Oliver and Macmahon of Cornell University, Van Vleck of Wesleyan University, Van Velzer of the University of Wisconsin, Beman and Ziwet of the University of Michigan, Smith of the University of Missouri, Fine of Princeton, Waldo of De Pauw University, Merriman of Lehigh University, Loud of Colorado College, Taber and Webster of Clark University, Ely of Vassar, and Messrs. Hulburt of Johns Hopkins University, Holgate of Northwestern University, and Blake of Columbia College.

[For most of the above details in regard to particular papers the writer is indebted to the courtesy of Prof. Klein.]

H. W. TYLER.

THE MEETING OF THE AMERICAN ASSOCIATION.

THE annual meeting of the American Association for the Advancement of Science was held at Madison, Wis., August 16–23, William Harkness of Washington presiding. In point of numbers the meeting was a disappointment, only 288 having finally registered; but in point of scientific interest and social intercourse it will rank among the most successful. The situation of Madison is especially favorable for the comfort of a gathering in the heated term. The citizens of the place were cordial in their reception of the Association and did much for its entertainment, while the excursion to the Dells of the Wisconsin on Saturday was a notably pleasant feature.

The attendance upon the meetings of the section in mathematics and astronomy was smaller than usual, but the proportion of papers of a distinctively mathematical character was comparatively great. The officers of the section were C. L. Doolittle, South Bethlehem, Pa., Vice-President; C. A. Waldo, Greencastle, Ind., Secretary; Geo. C. Comstock, Madison, Wis., Councilor; A. S. Flint, Madison, Wis., E. W. Hyde, Cincinnati, O., Malcolm McNeill, Lake Forest, Ill., together with Vice-President and Secretary, Sectional Committee; Frank H. Loud, Colorado Springs, Member of Nominating Committee; Rudolph Siemon, Fort Wayne, Ind.; F. W. Kuhne, Fort Wayne, Ind.; H. B. Hedrick, Washington, D. C., together with Vice-President and Secretary, Sectional Nominating Committee.

The following papers were presented:

1. Variations of latitude; Vice-Presidential address. C. L. Doolittle, So. Bethlehem, Pa.
2. A construction for the imaginary points and branches of algebraic loci. Frank H. Loud, Colorado Springs, Colo.
3. The screw as a unit in a Grassmannian system of the sixth order. E. W. Hyde, Cincinnati, Ohio.
4. Applications of the generalized logarithmic theorem. Alex. Macfarlane, Austin, Texas.
5. On the inscription of regular polygons. L. E. Dickson, Austin, Texas.
6. Latitude determination at Bethlehem, 1892–93. C. L. Doolittle, So. Bethlehem, Pa.
7. A determination of the constant of aberration by a modified form of the Loewy method. Geo. C. Comstock, Madison, Wis.
8. On the possibility of the algebraic solution of the general equation of the fifth degree. Mansfield Merriman, So. Bethlehem, Pa.
9. Space. S. S. Laws, Columbia, S. C.

No. 1 was a careful and thorough résumé of progress in the solution of the vexed problem of periodic and permanent latitude variations. To the final solution of this problem Mr. Doolittle has personally contributed data of great value by his persistent and arduous labors at Bethlehem.

No. 2 was a proposition worked out in considerable detail for the construction of a locus of two complex dependent variables.

No. 3 was a continuation of the author's labors in interpreting Grassmann's ideas, in modifying them and making them practical. The paper is a clear and simple discussion of the units that would arise and their relation to each other in a space popularly known as one of six dimensions.

In the absence of the authors a résumé of papers 4 and 5 was presented by the Secretary. Mr. Macfarlane has been working upon a modified Quaternion analysis. Some of his papers on this subject have appeared in recent volumes of the Proceedings of the American Association. The present paper applies one of his general results to a derivation, simpler and more direct than has heretofore been possible, of a number of important trigonometrical series. Mr. Dickson re-examined the subject of inscriptible polygons as presented in Gauss's works, and believes that he has simplified and made available Gauss's material besides adding much that is new.

No. 6 was in a sense a continuation of No. 1. No. 7 was presented to the section at the Washburn observatory, which was visited at Mr. Comstock's invitation.

No. 8 was in a sense a continuation of Mr. Merriman's paper of a year ago upon the equation of the third degree. The possibility suggested arises from an analogy in the geometrical construction of the roots of an equation of the fifth degree to those of an equation of the third and fourth degrees.

No. 9 was a metaphysical discussion.

No final decision was reached as to the place of the next annual meeting. The matter was left in the hands of a committee with the probability that Brooklyn will be chosen. If not Brooklyn, then some New England city. August 15–23 was set as the time. It is generally understood that in 1895 the Association will visit San Francisco. The President of the next meeting will be D. G. Brinton of Media, Pa.

The sectional officers chosen for 1894 are the following: Vice-President, Geo. C. Comstock, Madison, Wis.; Secretary, Wooster W. Beman, Ann Arbor, Mich.

C. A. WALDO.

NOTES.

A SPECIAL meeting of the NEW YORK MATHEMATICAL SOCIETY for the purpose of giving the members an opportunity of meeting Professor Felix Klein of Göttingen was held Saturday afternoon, September 30, at half past three o'clock, the president, Dr. McClintock, in the chair. The president introduced Professor Klein in a brief address describing Professor Klein's mission to America and paying tribute to him in his different capacities as teacher, investigator, organizer, and editor.

In his address to the Society, Professor Klein described a paper by Dr. Schilling* containing a non-euclidean development of spherical trigonometry in which the fundamental formulæ found a real geometrical interpretation when applied to a generalized spherical triangle whose sides and angles have complex values. Passing from the subject of non-euclidean geometry, he referred to other recent investigations connected with spherical trigonometry, mentioning a paper by himself † and one read by Dr. Study ‡ at the Chicago Congress. In concluding he deprecated the opinion obtaining among many persons that the so-called elementary subjects offer to the mathematician no opportunities for investigation.

This address was followed by a general discussion. Several questions were put to Professor Klein, to which he replied at length. After the adjournment of the meeting, those present were individually introduced to Professor Klein, and a general conversation ensued.

DURING the two weeks following the Mathematical Congress at Chicago, Professor Klein (as was announced in the BULLETIN, vol. II., p. 260) held a series of *colloquia* at Northwestern University, Evanston. His addresses at these meetings are in course of publication by Macmillan & Co. in commission for Northwestern University. T. S. F.

A COMPLETE edition of the works and lectures of Weierstrass is in course of preparation. As in the case of Kronecker's works, there will be two series. One of these, consisting of three quarto volumes, similar in style to the edition of Jacobi's works, will comprise all the published memoirs together with a number of papers not previously printed. The other series will contain Weierstrass's lectures, viz. : (1) two courses of lectures on elliptic functions, one more elementary, taking the differential equation as a starting point, the other based on the addition theorem ; (2) lectures on the applications of the elliptic functions ; (3) lectures on hyperelliptic functions ; (4) lectures on Abelian functions.

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* Mathematische Annalen, vol. 39, p. 598.

† Ibid., vol. 37, p. 573.

‡ See this number of the BULLETIN, p. 16.

The first volume of the memoirs is now in press. It opens with a paper on the theory of the elliptic functions presented by Weierstrass to the board of examiners of the Münster Academy, in 1841. This volume will probably be followed by the first part of the series of lectures.

Among the fall announcements we notice the following:

Ginn & Co. have in press a work by W. E. Byerly of Harvard University on "Fourier's series, and spherical, cylindrical, and ellipsoidal harmonics."

Macmillan & Co. announce of works by American authors: a "History of Mathematics," by Florian Cajori of Colorado College; a "Synthetic solid geometry," by N. F. Dupuis, of Queen's College, Kingston, Canada; a work on the "Resistance of ships and screw propulsion," by D. W. Taylor of the U. S. Navy Yard, Mare Island, California; and the second part (Statics) of A. Ziwet's "Mechanics." They also promise an English translation of Heinrich Hertz's "Researches on the propagation of electrical force," by D. E. Jones; and a new edition, revised and augmented, of J. J. Thomson's "Notes on recent researches in electricity and magnetism,"—the latter to appear early in 1894.

B. G. Teubner's announcements include, besides the edition of Kronecker's papers and lectures, a German translation, by H. Maser, of E. Goursat's lectures on partial differential equations ("Vorlesungen über die Integration der partiellen Differentialgleichungen erster Ordnung, gehalten an der Faculté des sciences zu Paris; bearbeitet von C. Bourlet"), and a work on elementary mathematics by G. Holzmüller ("Methodisches Lehrbuch der Elementar-Mathematik, im engsten Anschluss an die neuen Lehrpläne"), in two parts.

The second *Jahresbericht der deutschen Mathematiker-Vereinigung*, for the year 1892, has just appeared (Berlin, Reimer). It contains obituary notices on L. Kronecker (by H. Weber), H. Schröter (by Sturm), H. Gretschel (by Papperitz), J. Gierster (by Fricke); abstracts of the papers by Bjerknes, Fricke, F. Klein, F. Meyer, Schapira, Schlegel, Schumacher, and Study, that were to be read at the Nürnberg meeting planned for last year but abandoned on account of the cholera; finally, a more extensive report, "Die Entwicklung der Lehre vom Erddruck," by F. Kötter of Berlin.

As announced in the Bulletin, vol. II, p. 212, the third annual meeting of the *Deutsche Mathematiker-Vereinigung* was held at Munich, September 5th to 10th. The meeting took place in the great hall of the Polytechnic School, Professor Reye of Strassburg presiding. On the opening day there were present 74 members out of a total membership of 233. The exhibition of models, which filled four rooms of the School, was described and explained by its originator, Professor Dyck.

The preliminary announcement comprises two longer reports, viz, "On the theory of algebraic functions," by *Brill* and *Nöther*, and "Special chapters in mechanics," by *Henneberg;* and the following papers:

Bauschinger, Ueber Festigkeitsbestimmungen (mit Demonstrationen im mechanisch-technischen Laboratorium der technischen Hochschule);
H. Brunn, Zur Analysis situs;
K. Döhlemann, Zur Theorie des Nullsystems;
Freyberg, Ueber Töplers Vorlesungs-Apparat zur Statik und Dynamik starrer Körper.
Haas, Ueber einen Präcessionsglobus;
D. Hilbert, Zwei neue Beweise für die Zerlegbarkeit der Zahlen eines Körpers in Primideale;
E. v. Lommel, Zur Demonstration von Magnet-Kraftlinien;
R. Mehmcke, Ueber Rechenmaschinen und Rechenapparate;
F. Meyer, Ueber projective Gruppen der Ebene;
A. Pringsheim, Ueber die Taylor'sche Reihe;
H. Wiener, Weiteres über Grundlagen und Aufbau der Geometrie.

What is remarkable in this provisional programme is the very large share given to applied mathematics. This programme, and the additional fact of the simultaneous exhibition of models, instruments, and apparatus pertaining to pure and applied mathematics, would seem to indicate as the leading feature of the present meeting the great prominence given to the applications. The explanation will be found in the existence, at Munich, of a prosperous Polytechnic School maintaining close relations with the university and counting among its teachers the present energetic and active secretary of the German Mathematical Association, Professor W. Dyck. The coexistence and coöperation in the same city of university and engineering school, such as is found in Munich, is unique in Germany.

PROFESSOR F. LINDEMANN has been called from the University at Königsberg to Munich.

THE meeting of the French Association for the Advancement of Science was held this year at Besançon, August

3–13, Dr. Bouchard being president. The meeting next year will be held at Caen, with Mascart as president, and in 1895 at Bordeaux, with Trélat as president.

Professor E. W. Davis has accepted the professorship of mathematics at the University of Nebraska, Lincoln, Nebraska. T. S. F.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Aubic (A.). Les équations linéaires et leurs applications. [Thèse.] Paris, Gauthier-Villars, 1893. 4to. 81 pp. Fr. 3.50

Beau (O.). Einige Mitteilungen aus dem Gebiete der trigonometrischen Reihen und der Fourierschen Integrale. [Progr.] Sorau, 1893. 4to. 22 pp. Mk. 1.00

Björling (C. F. E.). Lärobok i Differentialkalkyl och algebraisk Analysis. 2d ed. Lund, 1893. 8vo. 12 and 366 pp. 1 plate. Mk. 12.00

Bowser (E. A.). An elementary treatise on analytic geometry, embracing plane geometry and an introduction to geometry of three dimensions. 17th ed., enlarged. New York, Van Nostrand, 1893. 12mo. Illustrated. Cloth. $1.75

Bricke (F.). Die kürzeste Linie auf dem einschaligen Hyperboloid. [Progr.] Grünberg, 1893. 4to. 16 pp. Mk. 1.00

Casey (J.). A treatise on the analytical geometry of the point, line, circle, and conic sections. 2d ed., revised and enlarged. 8vo. 584 pp. Dublin, Hodges, 1893. 12s.

Cranz (H.). Lehrbuch der analytischen Geometrie der Ebene. Erster Teil: Analytische Geometrie des Punktes und der Geraden. Mit einer Sammlung von Aufgaben, etc., nach System Kleyer. Stuttgart, Maier, 1892. 8vo. 7 and 266 pp. Mk. 6.00

Dalwigk (F. v.). Beiträge zur Theorie der Thetafunctionen von p Variablen. Leipzig, Engelmann, 1892. 4to. 43 pp. Mk. 2.00

Dauber (A.). Leitfaden der analytischen Geometrie. Helmstedt, 1893. 8vo. 35 pp. Illustrated. Mk. 1.20

Depène. Über die einem Dreieck ein- und umgeschriebenen Kegelschnitte. [Progr.] Breslau, 1893. 4to. 14 pp. Mk. 1.20

Descartes. See Tannery, P.

Dette (W.). Einleitung in die analytische Geometrie der Ebene. [Progr.] Elberfeld, 1893. 8vo. 76 pp. 4 plates. Mk. 1.50

Dodgson (C. L.). Curiosa mathematica. Part II: Pillow problems thought out during sleepless nights. London, Macmillan, 1893. 8vo. 2s.

Du Bois-Reymond (E.). Maupertuis. Leipzig, 1893. 8vo. 92 pp. 1 plate. Mk 1.50

Finsterbusch (J.). Beitrag zur synthetischen Geometrie ebener Kreissysteme und damit im Zusammenhange stehender höherer Kurven, Part III: Die Cartesischen Ovale und ihre konfokalen Büschel mit besonderer Berücksichtigung der dabei auftreden Konfigurationen und involutorischen Gebilde. [Progr.] Werdau, 1893. 8vo. 48 pp. Mk. 1.50

Fisher (F. H. G.). Ausgewählte Abschnitte aus einer synthetischen Theorie der Kegelschnitte. [Progr.] Leipzig, 1893. 4to. 24 pp. Mk. 1.00

Geigenmüller (R.). Elemente der höheren Mathematik, zugleich als Sammlung von Beispielen und Aufgaben aus der analytischen Geometrie, algebraischen Analysis, Differential- und Integralrechnung. 3d ed. Vol. I: Die analytische Geometrie. Mittweida, 1893. 8vo. 6 and 208 pp. 4 plates. Mk. 5.00

Gilbert (P.). See **Mansion**.

Goldschmidt (L.). Über den Satz Eulers $(1-x)(1-x^2)(1-x^3)\ldots = \sum_{0}^{\infty}(-1)^n \cdot \frac{3n^2 \pm n}{2}$. [Progr.] Gotha, 1892. 8vo. 12 pp.

Grassmann (H.). Anwendung der Ausdehnungslehre auf die allgemeine Theorie der Raumkurven und krummen Flächen. Part III: Krumme Flächen. 2d half. [Progr.] Halle, 1893. 4to. 31 pp. Illustrated. Mk. 1.80

Grzybowski (G.). O dotykaniach powierzchni spaczonéj obrotowéj. (On the contact of helicoidal surfaces.) [Progr.] Tarnopol, 1892. 8vo. 4 pp. 3 plates.

Hadaszczek (J.). Der Kegelschnitt in centrischer Collineation mit berührenden Kreisen. [Progr.] Ostrau, 1892. 8vo. 10 pp. 1 plate.

Haeusser (G.). Über die Fundamentaldeterminanten der allgemeinen homogenen Differentialgleichungssysteme. [Diss.] Heidelberg, 1892. 8vo. 32 pp. Mk. 1.20

Harkness (J.) and **Morley** (F.). A treatise on the theory of functions. New York, Macmillan, 1893. 8vo. 507 pp. Illustrated. $5.00

Heilermann (H.). Bemerkungen über die Verwendung der einfachsten Reihen bei der Begründung der algebraischen Regeln. [Progr.] Essen, 1893. 4to. 11 pp. Mk. 1.20

Heinitz (G.). Eine neue Bestimmung des quadratischen Restcharakters. [Progr.] Seesen, 1893. 8vo. 47 pp. Illustrated. Mk. 1.20

Huber (G.). Die Kegelfokalen. [Progr.] Bern, 1893. 4to. 56 pp. 3 plates. Mk. 1.80

Index du répertoire bibliographique des sciences mathématiques, publié par la Commission permanente du répertoire. Paris, Gauthier-Villars, 1893. 8vo. 14 and 80 pp. Fr. 2.00

Johnston (W. J.). An elementary treatise on analytical geometry. With numerous examples. New York, Macmillan, 1893. 12mo. 425 pp. Cloth. $2.60

Karagiannides (A.). Die nichteuklidische Geometrie vom Alterthum bis zur Gegenwart; eine historisch-kritische Studie. Berlin, 1893. 8vo. 44 pp. Mk. 1.80

Klein (F.). Einleitung in die höhere Geometrie, I. Vorlesung, gehalten im Wintersemester 1892-93; ausgearbeitet von F. Schilling. [Lithogr.] Göttingen, 1893. Small 4to. 8 and 566 pp. Mk. 7.50

Kloock (H.). Kritische Grundlegung der Arithmetik. Bonn, 1893. 8vo. 4 and 64 pp. Mk. 2.00

Klug (L.). A projectiv geometria etc. (Elements of projective geometry treated synthetically; Hungarian.) Budapest. 1892. 8vo. 272 pp. 14 plates. Illustrated. Mk. 4.00

Koch. Bestimmung der parallelen conjugirten Durchmesser zweier Ellipsoide. [Progr.] Siegburg, 1893. 4to. 18 pp. Mk. 1.00

LAMPE (E.). Die Entwickelung der Mathematik im Zusammenhange mit der Ausbreitung der Kultur. Berlin, 1893. 4to. 26 pp. Mk. 1.20

LAZZARI (G.). Trattato di geometria anilitica. Livorno, 1893. 8vo. 546 pp. Illustrated. Fr. 10.00

LEGENDRE (A. M.). Zahlentheorie. Nach der dritten Ausgabe ins Deutsche übertragen von H. Maser. 2 vols. 2d ed. Leipzig, Teubner, 1893. 8vo. Vol. I: 18 and 442 pp.; Vol. II: 12 and 453 pp. Mk. 12.00

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LACHLAN'S MODERN PURE GEOMETRY.

An Elementary Treatise on Modern Pure Geometry. By R. LACHLAN, M.A. Macmillan & Co., 1893. 8vo, pp. x. + 288.

IN this work, with slight exception, the field of operations is the plane. In the plane there are considered points, lines, and circles; and concerning these there is an excellent store of geometric facts, illustrating various methods. Of these methods so many are to be found in English works on plane geometry, whether they are called pure or analytic, that the delimitation is not very evident. It appears to be accepted as the fundamental distinction that in the books on pure geometry the treatment of curves by means of their equations is excluded. Whether co-ordinates are excluded is mainly a question of words; the algebra employed does not look like algebra, but this is of no serious importance; what is really out of bounds is the equation between variable co-ordinates, or the theory of the ternary form. With respect to the use of algebra, it can hardly be maintained, in the present case, that every equation ought to be associated in one's mind with a direct geometric concept; this would, in many cases, be a direct waste of energy, as, for instance, where trigonometry is employed. Since some algebra is permitted, the question arises whether a formal use of the algebra of binary forms is not proper in a work of this kind, and I shall briefly return to this point later.

In Dr. Lachlan's book, after three preliminary chapters, we have the explanation of harmonic ranges and pencils, and then a chapter on involution. The statement (page 40) that the double points of an involution can be imaginary, is misleading; it is better to say that they can be off the line in question. Chapter VI. gives an account of the triangle and of some of its more remarkable circles. Then follows (chapter VII.) the theory of four points or lines, or, as they are called after Townsend, tetrastigms and tetragrams. One hopes that more pleasing pet names will yet be adopted. This chapter contains some extensions which deal mostly with six points or lines. Among these may be noticed (page 99) the case of two triangles triply in perspective, or rather of three triangles, any two of which are in perspective as seen from any point of the third. A simple instance is when two equilateral triangles have the same centre; the centres of perspective form a third equilateral triangle concentric with the others.

The theory of perspective is contained in chapter VIII. The symmetry of the diagram on page 100 should be remarked, as is done by Reye in the preface to his "Geometrie der Lage." It is a configuration of 10 points and 10 lines; there are 3 points on each line and 3 lines through each point. Taking 5 points in space, we have 10 lines through 2 points and 10 planes through 3 points, and the configuration is cut out of any plane by these lines and planes. The chapter includes an account of Pascal's theorem and of some of its developments.*

In chapter IX. is the theory of similar figures, which appears to be one of the most important outcomes of the work recently bestowed on the triangle. We have then (chapter X.) the theory of pole and polar with regard to a circle, of conjugate triangles, of four points of a circle, and of four tangents. Chapter XI. gives the theory of reciprocation, the next chapter that of two circles, and chapter XIII. that of coaxial circles, with a valuable discussion of Poncelet's theorem. The idea of the porism, incidentally mentioned on page 215, might have been introduced in other places; for example, in the diagrams of pages 100 and 164 poristic properties lie unconcealed.

Chapter XIV. deals with inversion. One important distinction between the theories of inversion and projection ought to be brought out. To suit the theory of projection we regard ∞ as a *line;* but for similar reasons ∞ is a *point* in the theory of inversion.

The next chapter on systems of circles contains much new matter. Among other things the triangles formed by three circles not meeting in a point receive here a fuller treatment than they are used to. The author makes use of the work of Mr. A. Larmor, and much also is no doubt his own property. Why does he not refer us to his own papers, in particular to the memoir in the *Philosophical Transactions,* vol. 177?

The last chapter contains the theory of cross-ratios. These are defined trigonometrically for the rays of a pencil, as was done in the harmonic case. There seems no sufficient reason for putting so fundamental an invariant as the cross-ratio so late.

The book contains many examples for solution. We welcome the author's statement that in all cases the solution rests on propositions immediately preceding, for exactly what the beginner needs, when he thinks he understands some theory, is a special application—an example and not a problem. We could wish that the names of the responsible parties were added to the examples in more cases; the practice of attaching a name to a result, even if the result be trifling, has become common of late, and is apt to stimulate those who fight *pro areis et focis*.

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* It may not be out of place to refer to Mr. Richmond's paper on Pascal's hexagram, *Cambridge Phil. Trans.* vol. XV., pp. 267–302.

Mr. Matthew Arnold in his "Science and Literature" gives a canon for judging a work of art: "Fit details strictly combined, in view of a large general result nobly conceived." The present book would avoid condemnation under this canon if some remarks were made at the outset on the principle of correspondence. We should then see that in projection we are studying a one-to-one correspondence of point to point, or of line to line; in inversion a different one-to-one correspondence of point to point, in which a circle answers to a circle, and the line appears as a special circle; in reciprocation a one-to-one correspondence of point and line; and we should be prepared for the direct correspondence of point and circle, which is discussed in chapter xv.—this apparently novel correspondence being, however, two-to-one. Lastly, we should see that involution belongs both to the projective theory and to that of inversion.

Throughout the book metrical methods are freely employed; in fact, there is little of the ideas of the *reine Geometrie der Lage*, and the works of the constructors of that edifice are not so much as referred to. Herein is a manifest injustice to the student, seeing that a mere note would have sufficed to put him on the track.

On the other hand, it seems a pity to make no use of the idea of the stroke or vector, when this would involve merely a modification of those proofs which are based on geometric constructions. Instances of important geometric ideas thus passed over, but attainable without any greater use of algebra, are those of involution as applied to points of a plane, and of the cross ratios of four points.

It seems not unlikely that a work of this kind is the proper place for the systematic development of the geometric meaning of the elements of the theory of binary forms. Beginning with the range of points, let us measure from a selected point, replace AB by $b-a$, and so forth. We then handle the quadratic equation, and deal with involutions and harmonic pairs as in Salmon's "Conic Sections"; using, however, geometric constructions freely. Let us then bear in mind that a and b can be complex, and make the necessary modifications in our diagrams. Sooner or later the single point of reference is replaced by two fixed points, and the determination of any point is effected by assigning the ratio of the strokes to it from the two points. We have then homogeneous co-ordinates. Proceeding to the cubic, or triangle of points, we should naturally put in the foreground the covariants—the polars of a point, the Hessian pair, the Jacobian triangle; and we should have at hand a scientific theory which covers many of the facts recently added to our knowledge of the triangle, and which indicates many others. A similar treatment of the quartic would follow. It is hardly an objection to say that the reader is not supposed to know about covariants, for here is as good a way as any to introduce them to him. The plan thus briefly sketched has, of course, no pretension to novelty; its claim to consideration here lies in its ability to classify and connect much of that matter of the book which is not already connected and classified by the methods of projection and reciprocation.

I have dwelt enough on defects in the author's programme. But in its handling so much power is displayed that one hopes he will set his hand to the plough again, and deal largely with some part of the present outlook as stated in Klein's "Vergleichende Betrachtungen" (BULLETIN, July, 1893); throwing examination schedules, if they interfere, to the four winds of heaven.

FRANK MORLEY.

WEST FALMOUTH, MASS., *August*, 1893.

PAPERS OF MR. CHARLES CHREE ON VORTICES IN A COMPRESSIBLE AND ROTATING FLUID.

THE recent appointment of Mr. Charles Chree, M.A., of King's College, Cambridge, Eng., to be Superintendent of Kew Observatory as the successor of the late Mr. Whipple, leads us to hope that the study of the mathematical and physical problems whose solution is so important to the progress of meteorology will now receive a great stimulus in England. Mr. Chree has hitherto been known to us mostly through his excellent works in pure mathematics as applied to the subjects of elasticity and of vortex motion: his papers on vortex motion are probably but little known to the mathematical physicists of America, and with the author's help we are able to present the following account of his memoirs, which will be of interest to meteorologists in so far as the assumptions which underlie his mathematical solutions harmonize with the conditions that prevail in the atmosphere. On this latter point Mr. Chree very properly and modestly says:

"I should be sorry if any one supposed I profess to have actually solved the exact problems presented by nature. The lay reader is so much at the mercy of the mathematician that I think the latter is taking a most unfair advantage of his position if he avoid taking a reasonable care to prevent the general public from regarding his hypothetical problems as actual facts."

This most admirable statement of a fundamental principle of scientific integrity is followed by a paragraph repeating in detail the cautions that are interspersed throughout Mr. Chree's papers as to the practical application of his work to the earth's atmosphere:

"Vortices on the earth do not extend to infinity; their cross-section is often not small compared to their distance apart; their diameter is often not *very* small compared to the earth's radius; they are often not approximately circular in section; there are vertical currents, and the horizontal velocity is not the same either in magnitude or direction at all points in the same vertical line; air, moist or dry, is not wholly destitute of viscosity, etc., etc.; there are thus enormous obvious differences between the simple mathematical problems of my papers and the actual state of matters on the earth; and if there is any resemblance between the results in the papers and actualities, it may quite as likely be a pure accident as not."

There are so many kinds of vortex motion present in the atmosphere, from the minute streams of hot air that rise from every point of the landscape when bathed in sunshine, or the columns of smoke that ascend from every chimney, or the rolling front of a horizontal flow of air, up to the cumulus cloud, the waterspout, the tornado, the hurricane, and even the general circulation of the atmosphere, and these present such a great variety of characteristic differences, that it will not be altogether a pure accident if the mathematician in solving some simple problem has paved the way for a better understanding of the complex phenomena that the meteorologist has to study.

Mr. Chree's papers deal with vortex motion in compressible fluids. The exact mathematical problems he has treated are those of infinite straight vortices or of circular ring vortices. The former case includes a straight vortex perpendicular to an infinite plane which limits it in one direction. In his first paper, "On vortices" (*Proc. Edinb. Math. Society*, vol. v., session 1886–87), it is shown that if there be two parallel thin straight vortices in a compressible fluid limited by infinite planes, then their distance apart, if their density alters, does not remain constant—as in a liquid—but increases or diminishes according as the sum of their cross-sections is increasing or diminishing. The path of one such vortex relative to the other is an equiangular spiral. On pages 58, 59 are pointed out certain conclusions that would hold in the case of cyclones on the earth *if* the conditions sufficiently resembled those of the problem.

A second paper, "Vortex rings in a compressible fluid" (*Proc. Edinb. Math. Soc.*, vol. VI., pages 59–68), works out the case of a single ring or of a ring and its image in an infinite plane—which might apply to a ring outside a sphere of *large* radius—the plane of the ring being parallel to the infinite plane. There is no "practical" application in that paper.

Mr. Chree's third paper, "On vortex motion in a rotating fluid" (*Proc. Edinb. Math. Soc.*, vol. VII., pages 29–41), deals with questions more nearly allied to the actual problems on the earth. The fluid is supposed to rotate with uniform angular velocity round an axis. He first solves the case of a vortex and its image in an infinite plane. Then comes the case of a vortex inside, but not at the centre, of a right circular cylinder, the whole system rotating round the cylinder's axis uniformly. This requires an image system of course. The conclusions the results would lead to, if applicable to a vortex on the earth's surface, are pointed out in pages 39–41. Some of them seem to have some resemblance to actual phenomena.

His last published paper, "On equations of vortex motion with special reference to the use of polar co-ordinates" (*Proc. Edinb. Math. Soc.*, vol. VIII., pages 43–64), deals generally with polar co-ordinates, and treats the case of a vortex ring in a rotating fluid, outside of a sphere, or parallel to an infinite plane. There would *appear* to be a motion of the ring similar to that of the plane of vibration of a Foucault's pendulum, the angular velocity varying as the sine of the latitude. There is, however, a difficulty in satisfying the conditions at the surface of the ring. This perhaps might be got rid of by supposing the existence of viscosity in the fluid.

Between the publication of the first and second of the above papers Mr. Chree published in the *Messenger of Mathematics*, November, 1887, a paper "On vortices in a compressible fluid," which may be considered as an extension to compressible fluids of Professor J. J. Thomson's prize-essay "On the motion of vortex rings;" it treats of the vibrations and changes of shape of cylindrical vortices in a compressible fluid.

In the four first-mentioned papers Mr. Chree gives innumerable hints as to motions in the earth's atmosphere, thus showing that he has meteorological problems in his mind, although he disclaims having as yet resolved any questions except those that relate to a simplified ideal atmosphere.

We trust that the strictly meteorological work of Mr. Chree's new position will tempt him to apply modern mathematical analysis to the winds, the clouds, and the storms of the actual atmosphere.

CLEVELAND ABBE.

PICARD'S TRAITÉ D'ANALYSE.

Traité d'Analyse. By ÉMILE PICARD. Vol. I., 1891, pp. XII. + 457; Vol. II., 1893, pp. XIV. + 512. Paris, Gauthier-Villars.

ONE of the ablest of American mathematicians said to the writer not long ago, "we have waited fifty years for this book"! While neither the speaker's age nor the state of mathematical analysis would warrant one in taking this statement literally, it nevertheless expressed a feeling which must have been experienced during recent years by every student of analysis. A great many treatises having the above title, or one differing but little from it, have appeared, particularly in France and Germany, during the last twenty-five years or thereabouts, many of them good, some of them excellent—as for example Jordan's "Cours d'Analyse," while some had perhaps no really good reason for existence. None of these treatises though, however valuable they may have been, have filled or even attempted to fill the place which will be occupied by Picard's "Traité d'Analyse."

The necessity for a treatise on analysis which should present the subject from the modern point of view has for several years been most obvious. The extraordinary developments in the theory of functions, in differential equations, and in certain purely algebraical theories, and the important applications of the results of these developments to geometrical, physical, and astronomical problems, have made such a treatise almost indispensable. The difficulties caused by the magnitude and complexity of the subject might well deter any one from undertaking to give an account of it, so that any mathematician, whatever his merit, would probably be thanked if he had made a fair attempt in that direction. When, however, such a mathematician as M. Émile Picard undertakes the task more than ordinary gratitude is due. M. Picard is one of the very first analysts of the age, both as an original investigator and in virtue of the vast range of his knowledge; joined to these claims to be considered the proper person to write the Traité d'Analyse of to-day, he possesses a most remarkably clear and elegant style of presenting a subject, whether it be in the form of a memoir embodying the results of his own personal researches or in the form of a lecture or chapter containing an account of the researches of others. The elegance and conciseness of M. Picard are not, however, at the expense of rigor, as every reader of his work is aware, and as every student who has had the pleasure of listening to his lectures can testify.

The first volume of M. Picard's work is in a measure an introductory volume, and is principally concerned with the development of comparatively elementary subjects, though some of the subjects treated and the general method of development would hardly find place in a treatise for beginners. There is in many places such a close relation between the subjects treated in vol. I. and vol. II., that it was thought better to notice these two volumes together. Vol. I. is divided into three parts: Part First, Simple and multiple integrals; Part Second, Laplace's equation and its applications; developments in series; Part Third, Geometrical applications of the infinitesimal calculus. Vol. II. contains seventeen chapters. Fifteen deal with functions of a complex variable, further developments in series, conform representation, Dirichlet's problem, roots common to two simultaneous equations (also discussed in vol. I.), integrals of non-uniform functions, algebraic functions of one variable, Riemann's surfaces, and Abelian integrals. There is one chapter on general theorems concerning differential equations, and one on the applications of these theorems. A word is necessary as to the author's primary intention and the reasons which caused him to modify it. In the introduction to vol. I. he says: "En publiant ce *Traité d'Analyse*, j'ai pour but principal de développer la partie de mon cours de la Faculté des Sciences, relative à la théorie des équations différentielles. Cet ouvrage sera donc surtout un traité général sur la théorie des équation différentielles à une ou plusieurs variables. Je n'ai cependant pas cru devoir adopter ce dernier titre, et cela pour deux raisons." The first of these reasons concerns principally M. Picard's students, and need not be cited. "Un autre motif, d'un caractère tout scientifique, m'engageait encore à garder le titre un peu vague de *Traité d'Analyse;* c'est que la théorie des équations différentielles est intimement liée a plus d'une autre théorie qu'il nous faudra approfondir. Pour ne citer qu'un exemple, l'étude préliminaire des fonctions algébriques est indispensable, quand on veut s'occuper de certaines classes d'équations différentielles. Nous ne nous bornerons donc pas strictement à l'étude des équations différentielles; nous rayonnerons autour de ce centre." In the introduction to vol. II. we find the following statement: "J'avais annoncé dans le premier volume que je comptais m'occuper surtout dans ce Traité de la théorie des équations différentielles. On trouvera ici un seul chapitre consacré à cette théorie telle qu'on l'entend ordinairement dans les ouvrages classiques. Je pourrais prétexter que l'équation de Laplace est une équation différentielle; j'aime mieux avouer que mon plan s'est un peu élargi. Je m'occuperai particulièrement, dans le tome III., de l'étude des équations différentielles, mais je n'oserais pas affirmer cependant que je n'aurais pas encore plusieurs parenthèses à ouvrir." As an indication of a most important subject to be treated later, the writer may perhaps be permitted to quote a line from a private letter: "... Les fonctions fuchsiennes, hyper-fuchsiennes etc. ne seront pas oubliées dans la rédaction de mon traité." That is certainly pleasant news for the student of those functions.

Chapter I. of vol. I. is concerned with definite integrals. The opening words of the chapter may be quoted: "Le calcul intégral a pris naissance le jour où l'on s'est posé la question suivante: une fonction $f(x)$ étant donné existe-t-il une fonction qui admette $f(x)$ pour derivée, c'est-à-dire une fonction telle que l'on ait

$$\frac{dy}{dx} = f(x).$$

On a d'abord répondu à cette question par une représentation géométrique qui n'a aucune valeur par elle-même, mais qui n'en a pas moins fait faire de grands progrès à la science. On construisit la courbe $y = f(x)$, et l'on considérait l'aire comprise entre cette courbe, l'axe des x et deux parallèles à l'axe des y, l'une fixe, l'autre variable; on montrait que l'aire, considérée comme fonction de l'abscisse x de cette dernière ordonnée, est une fonction de x ayant $f(x)$ pour dérivée. Il est clair qu'à moins d'admettre que la notion d'aire est une notion première, il n'y a pas là une réponse rigoureuse au problème posé."

The author proceeds now to give a precise meaning to the notion of a definite integral in the case when the function to be integrated, $f(x)$, is continuous between the limits within which the variable x is restrained to lie. Admitting that there exists a function y satisfying the equation

$$\frac{dy}{dx} = f(x),$$

and taking the value y_0 for $x = a$ and the value Y for $x = b$; divide the interval (a, b) into n intervals given by the values $x_1, x_2 \ldots x_{n-1}$, and let $y_1, y_2 \ldots y_{n-1}$ be the corresponding values of y. If the interval $x_1 - a$ is small enough, the quotient

$$\frac{y_1 - y_0}{x_1 - a}$$

differs but little from $f(a)$, and we can write the approximate relations

$$y_1 - y_0 = (x_1 - a)f(a);$$

$$y_2 - y_1 = (x_2 - x_1)f(x_1);$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Y - y_{n-1} = (b - x_{n-1})f(x_{n-1}).$$

Adding these, we get

$$Y - y_0 = (x_1 - a)f(a) + (x_2 - x_1)f(x_1) + \ldots + (b - x_{n-1})f(x_{n-1});$$

an approximate equality, which will be more and more accurate as the number of intervals is increased, each of them becoming smaller and smaller. A preliminary lemma is now established where by the *oscillation* of a continuous function in an interval is meant the difference between the greatest and the least values which it takes in the interval. The lemma is as follows: Suppose an interval (a, b) where to fix the ideas $a < b$ and a function $f(x)$ continuous in this interval. Having given a positive number ϵ as small as we please, *we can always find a positive quantity* δ, *such that in every interval contained in* (a, b) *and less than* δ *the oscillation of the function shall be less than* ϵ.

This lemma established, the author proceeds to the demonstration of the following fundamental theorem. *The sum*

$$(x_1 - a)f(a) + \ldots + (b - x_{n-1})f(x_{n-1})$$

tends towards a limit, when all the intervals (x_{i+1}, x_i) *tend towards zero according to any law whatever at the same time that their number increases indefinitely.* Following the establishment of this theorem come some geometrical applications to the areas and lengths of arcs of curves, and then the notion of integration by parts is introduced. Under this head Picard gives an important formula of Kronecker's. Let $f(x)$, $g(x)$ be any two functions of x, and let as usual $f^{(n)}(x)$ denote the nth derivative of $f(x)$, and let $g^{(n)}(-x)$ denote the nth derivative of $g(x)$ when after differentiation x has been replaced by $-x$. Start from the identity

$$f^{(h)}(x)g^{(n-h)}(-x) - f^{(h-1)}(x)g^{(n-h+1)}(-x)$$

$$= \frac{d}{dx}[f^{(h-1)}(x)g^{(n-h)}(-x)].$$

Making successively $h = 1, 2, \ldots, n$, and adding, we have

$$f^{(n)}(x)g(-x) - f(x)g^{(n)}(-x)$$
$$= \sum_{h=1}^{h=n} \frac{d}{dx}[f^{(h-1)}(x)g^{(n-h)}(-x)];$$

integrate now between a and b, and we have

$$\int_a^b f^{(n)}(x)g(-x)dx - \int_a^b f(x)g^{(n)}(-x)dx$$
$$= \sum_{h=1}^{h=n} \left[f^{(h-1)}(x)g^{(n-h)}(-x)\right]_a^b. \qquad (\alpha)$$

The calculation of the second integral is thus conducted to that of the first. This formula of Kronecker's was given in 1884, and has not before appeared in any treatise on analysis, so far as the writer is aware. It is capable of many interesting applications. Picard gives two applications. First writing

$$f(x) = F'(x), \qquad g(x) = \frac{(x+b)^n}{1 \cdot 2 \ldots n},$$

we get

$$F(b) = F(a) + (b-a)F'(a) + \ldots$$
$$+ \frac{(b-a)^n}{1 \cdot 2 \ldots n} F^{(n)}(a) + \int_a^b F^{(n+1)}(x)\frac{(b-x)^n}{1 \cdot 2 \ldots n}dx,$$

which is Taylor's theorem. The remainder here presents itself in the form of a definite integral, but is readily changed into the ordinary form. Again, writing

$$g(x) = (x+a)^n(x+b)^n,$$

and letting $f(x)$ denote an arbitrary polynomial of degree $n-1$, we find from Kronecker's formula (α)

$$\int_a^b f(x)P_n(x)dx = 0,$$

where

$$P_n(x) = A\frac{d^n(x-a)^n(x-b)^n}{dx^n}, \qquad (\beta)$$

A denoting a constant. The relation (β) is shown to define completely the polynomial P_n to a constant factor *près*—this polynomial is, of course, Legendre's polynomial—i.e., a zonal spherical harmonic.

The case of the change of variable in the integral is now considered, and then the notion of a definite integral is extended to the case where one limit becomes infinite. Here Cauchy's rule relative to the convergence of series furnishes a natural and interesting application, which, however, need not be quoted. The conditions under which we may differentiate under the sign $\int$ are now carefully considered, and as a consequence of this method of calculating an integral the formula

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

is found. This chapter closes by an extension of the notion of a definite integral to the case of complex functions of a real variable. If $F(x)$ is a real function of x, and if $\psi(x)$ is a positive function between a and b, we know that

$$\int_a^b F(x)\psi(x)dx = F(\xi)\int_a^b \psi(x)dx,$$

where ξ is a value of x between a and b. Darboux has given a formula analogous to this in the case where the function $F(x)$ is of the form $f(x) + i\phi(x)$, x being a real variable. Write

$$I = \int_a^b F(x)\psi(x)dx,$$

where $F(x) = f(x) + i\phi(x)$ and where $\psi(x)$ is positive between a and b. Darboux's formula is

$$I = \lambda F(\xi)\int_a^b \psi(x)dx,$$

where ξ is a value of x lying between a and b, and λ is a quantity whose modulus is at most equal to unity. From this formula we derive readily the extension of Taylor's theorem to the case where $F(x)$ has the above form.

Chapter II. is devoted to indefinite integrals. It begins with a brief and elegant account of the integrals of rational fractions, and then passes on to hyperelliptic integrals, and so on to the integrals of algebraic differentials in general.

The hyperelliptic integrals are shown to reduce to the two types

$$\int\frac{f(x)dx}{\sqrt{R(x)}},\quad \int\frac{\phi(x)dx}{(x-a)^{\alpha}\sqrt{R(x)}},$$

where $f(x)$, $\phi(x)$, and $R(x)$ are polynomials and α is a positive integer. The cases where $R(x)$ is of odd degree or of even degree are seen to be conducted the one to the other, so that $R(x)$ is supposed at once to be of degree $2p+1$. It is now proved that there exist $2p$ integrals of the first type, and of the form

$$\int\frac{x^{\mu}}{\sqrt{R(x)}}dx,\qquad \mu=0,1,2,\ldots(2p-1),$$

and then that among these there are p integrals of the *first kind*, that is, integrals which remain finite when x increases indefinitely, the remaining integrals of this type being of the *second kind.*

Passing now to integrals of the second type,

$$\int\frac{\phi(x)dx}{(x-a)^{\alpha}\sqrt{R(x)}},$$

it is seen to be necessary to distinguish between the two cases when a is and is not a root of $R(x)=0$; and it is finally shown that the integrals of the second type conduct to those of the first type and to integrals of the form

$$\int\frac{dx}{(x-a)\sqrt{R(x)}},$$

where a is not a root of $R(x)=0$.

The consideration of the hyperelliptic integrals includes as a particular case the elliptic integrals, and the case where $R(x)$ is a polynomial of degree two.

We pass on now to the case of the integrals of algebraic differentials in general, *i.e.*, to the case of the Abelian integrals. These are all comprised in the form

$$\int F(x,y)dx,$$

where F is a rational function of x and y, and where x and y are connected by the irreducible algebraic equation $f(x,y)=0$. The curve $f=0$ is supposed of degree m, and, further, the aggregate of all the homogeneous terms of degree m is supposed made up of m distinct linear factors, none of which reduce to either x or y, which comes to saying geometrically that the m asymptotic directions of the curve f are distinct, and none of them are parallel to the axes of x or y. This is, of course, a permissible supposition, as such a condition can always be realized by a homographic transformation.

The above integral is now conducted by a known and purely algebraic reduction to the two types

$$\int \frac{P(x,y)dx}{f'_y}, \int \frac{Q(x, y)dx}{(x-a)^\alpha f'_y},$$

where P and Q are polynomials in x and y and α is a positive integer; and, finally, it is here shown that in the integrals of the first type the degree of the polynomial $P(x, y)$ can be lowered to $2m - 4$.

In the case of the integrals of the second type two cases have to be considered, according as the straight line $x - a = 0$ meets the curve $f = 0$ in m distinct points, or is tangent to the curve. Again, the curve $f = 0$ may or may not have singular points; if, however, the curve have only double points with distinct tangents, it is shown that under both of the above hypotheses as to the line $x - a = 0$ the integral

$$\int \frac{Q(x, y)dx}{(x-a)^\alpha f'_y}, \quad \alpha > 1,$$

can be conducted to an analogous integral, in which the exponent of $x - a$ is equal to unity, say

$$\int \frac{R(x, y)dx}{(x-a)f'_y},$$

when R is a polynomial in (x, y). This can finally be reduced by aid of the equation $f(x, y) = 0$ to the form

$$\int \frac{R(y)dx}{(x-a)f'_y},$$

where R is a polynomial in y alone of the degree $m - 1$ at most. This chapter closes by a brief account of Hermite's researches on the integration of rational functions of sin x and cos x.

The subject of the Abelian integrals is resumed in chapter XIV., of vol. II., after having devoted one chapter, XIII., to the general properties of algebraic functions of one variable, a theorem of Nöther's, and Riemann's surfaces. It will perhaps be as well to notice these chapters in this place rather than wait until they occur in the course of going over the two volumes chapter by chapter. In chapter XIII. the author begins by defining an algebraic function, and obtains its development in the region of a point. These familiar results are presented in an exceedingly simple and elegant form. Let $f(u, z) = 0$ be the irreducible equation defining the function u, and suppose it to be of the degree m in u. Introducing now the notion of *critical points*, and making use of the known theorem (established in vol. I.) that if this equation has for $z = \alpha$ n roots equal to β, it will have for z near α n roots and n only near β, the following fundamental theorem of algebraic functions is established: *The roots which for $z = \alpha$ become equal to β form one or several circular systems, and the roots of any one circular system are, in the region of α, represented by a development of the form*

$$u = \beta + A(z - \alpha)^{\frac{1}{n'}} + B(z - \alpha)^{\frac{2}{n'}} + \ldots$$

To the n' determinations of $(z - \alpha)^{\frac{1}{n'}}$ correspond n' values of u, and these are the values which permute circularly around the point α.

This result is sufficient for the general theory of algebraic functions and their integrals, but practically we need to know how to obtain the different circular systems and the corresponding numbers n'. A very brief, but sufficient, account is here given of Puiseux's researches in this direction.

Following these classical investigations on circular systems comes a most welcome account of the important theorem of Nöther's, which shows that the generality of the theory of algebraic functions is in no wise lessened if we confine ourselves to curves having no other singular points than *multiple points with distinct tangents*. The importance of this theorem, which was given by Nöther in vol. IX. of the *Mathematische Annalen*, is at once seen by any one who has studied even a very little of the theory of algebraic functions. The demonstration given of this theorem is due to M. Simart, and is of such interest that it deserves to be reproduced here.

Let $f(x, y) = 0$ be the equation of an algebraic curve of degree m, and let the origin be a multiple point of order n on the curve, that is, having n tangents. This equation can be put in the form

$$\phi_n(x, y) + \phi_{n+\lambda}(x, y) + \ldots + \phi_m(x, y) = 0, \qquad (E)$$

and we shall admit that aside from the origin the axes of x and y only meet the curve in simple points, that the tangents at the origin do not coincide with the axes, and, finally, that the asymptotic directions are distinct and different from the directions of the axes. These hypotheses granted, make the substitution $y = \frac{x}{Y}$. To the curve (E) will correspond point by point the curve (E') of degree $2m - n$, viz.,

$$F(Y, x) = Y^{m-n}\phi_n(Y, 1) + x^{\lambda} Y^{m-n-\lambda}\phi_{n+\lambda}(Y, 1) + \\ \ldots + x^{m-n}\phi_m(Y, 1) = 0. \qquad (E')$$

To a multiple point of order n' of the curve (E), not on the axis of x or on that of y, corresponds evidently a multiple point of order n' of the curve (E'). To the value $x = 0$ on the first curve correspond n values of y which are zero, and $m - n$ other distinct values. To these points there correspond in the second curve first a multiple point of order $m - n$ $(x = 0,\ Y = 0)$ with $m - n$ distinct tangents, since for $x = 0$ the $m - n$ corresponding values of y are different from each other and different from zero; and again, n simple or multiple points of the axis of y determined by

$$\phi_n(Y, 1) = 0.$$

To the value $x = \infty$ correspond m distinct values of Y; further, for $Y = \infty$ the $m - n$ values of x are distinct.

We have then substituted for the curve (E) a curve (E'), which, outside of the axis of x, has the multiple points of the first; but which has at the point $x = 0$, $Y = 0$ a multiple point of order $m - n$ with distinct tangents, and upon the axis of Y $(x = 0)$, n points determined by $\phi_n(Y, 1) = 0$, which may be distinct or coincident.

Let us suppose that the equation

$$\phi_n(Y, 1) = 0$$

has a multiple root $Y = Y_1$, to which corresponds a multiple point of order $n' \leqq n$. By an arbitrary homographic transformation

$$x = \frac{ax' + by'}{a''x' + b''y' + c''}, \qquad Y - Y_1 = \frac{a'x' + b'y'}{a''x' + b''y' + c''},$$

we can substitute for the curve (E') the curve (E_1) of degree $2m - n$, viz.:

$$\phi'_{n'}(x', y') + \phi'_{n'+\lambda'}(x', y') + \ldots + \phi'_{2m-n}(x', y') = 0. \quad (E_1)$$

This curve satisfies the same conditions as the curve (E), and its asymptotic directions are distinct and different from zero. It has, aside from the points determined by $\phi_n(Y, 1) = 0, x = 0$, multiple points corresponding to the multiple points of (E), plus: *one* multiple point of order $m - n$ with distinct tangents arising from the point $x = 0$, $Y = 0$; *one* multiple point of order $m - n$ with distinct tangents arising from the point $Y = \infty$; and, finally, *one* multiple point of order m with distinct tangents arising from the point $x = \infty$.

We can now apply to the curve (E_1) the same transformations as to the curve (E), and continue until we arrive at an equation such as $\phi_n(Y, 1) = 0$, all of whose roots are distinct. *This series of operations with certainty come to an end;* for, suppose that we have constantly $n' = n$. The curve (E_1) of degree $m_1 = 2m - n$ has multiple points corresponding to those of the curve (E) plus *two* multiple points of order $m - n$ with distinct tangents equivalent to $(m - n)(m - n - 1)$ double points, and *one* multiple point of order m with distinct tangents equivalent to $\frac{m(m-1)}{2}$ double points. The curve (E_2) of degree $m_2 = 2m_1 - n$ differs from the curve (E_1) in the same way that (E_1) differs from (E).

Finally, denoting by d_k the number of double points of the curve E_k corresponding to the multiple points with distinct tangents which have been successively introduced, we shall have

$$d_k = \sum_{i=0}^{i=k-1} \left[(m_i - n)(m_i - n - 1) + \frac{m_i(m_i - 1)}{2} \right].$$

Now we have

$$m_i = 2^i(m - n) + n;$$

hence

$$d_k = \frac{3}{2}(m - n)^2 \left(\frac{2^{2k} - 1}{2^2 - 1} \right)$$
$$+ (m - n)\frac{2n - 3}{2}(2^k - 1) + k\frac{n(n - 1)}{2}.$$

The difference between this number and the maximum number of double points of an irreducible curve of order m_k, viz.,

$$\frac{(m_k - 1)(m_k - 2)}{2},$$

is

$$-\frac{(m-n)^2}{2} - (m-n)\frac{2n-3}{2} + k\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2}.$$

This difference would become positive if k were sufficiently large; but this is impossible: it is therefore necessary that n diminish until it becomes equal to unity. A time will come when the multiple point at the origin of the primitive curve will have been replaced in the transformed curve by multiple points of a lower order, and during this transformation we shall only have introduced multiple points with distinct tangents. Proceeding thus step by step, *we shall finally arrive at a curve which has only multiple points with distinct tangents.*

After a few words on birational transformations Picard enunciates Nöther's theorem in the following form: *We can always, by a Cremona transformation, transform any algebraic curve whatever into another having only multiple points with distinct tangents.*

It is next shown that (leaving aside a certain difficulty which need not be mentioned here) we can by certain transformations always arrive at a curve having only double points with distinct tangents. Finally, if α_i denote the number of multiple points of order i with distinct tangents, and w denote the number of points of ramification (that is, values of x for which *two* values of y from the equation $f(x, y) = 0$ permute), we arrive at the formula

$$w = m(m-1) - \Sigma\alpha_i i(i-1),$$

the sum Σ being relative to the different values of $i (i \geqq 2)$.

This chapter concludes with a rather brief but very satisfactory section devoted to Riemann's surfaces. Very few words are necessary in speaking of this section of the Traité, but these may be preceded by another quotation from the Introduction: "Un chapitre traite des surfaces de Riemann, dont l'étude a été laissée un peu trop de côté en France; on peut, par une représentation géometrique convenable, rendre intuitifs les principaux résultats de cette théorie. Cette vue claire de la surface de Riemann une fois obtenue, toutes les applications se déroulent avec la même facilité que dans la théorie classique de Cauchy relative au plan simple. Mais il importe de juger à sa véritable valeur la belle conception de

Riemann. Ce serait une vue incomplète que de la regarder seulement comme une méthode simplificative pour présenter la théorie des fonctions algébriques. Si importante que soit la simplification apportée dans cette étude par la considération de la surface à plusieurs feuillets, ce n'est pas là ce qui fait le grand intérêt des idées de Riemann. Le point essentiel de sa théorie est dans la conception *a priori* de la surface connexe formée d'un nombre limité de feuillets plans, et dans le fait qu'à une telle surface conçue dans toute sa généralité correspond une classe de courbes algébriques. Nous n'avons donc pas voulu mutiler la pensée profonde de Riemann, et nous avons consacré un chapitre à la question difficile et capitale de l'existence des fonctions analytiques sur une surface de Riemann arbitrairement donnée; le problème même est susceptible de se généraliser si l'on prend une surface fermée arbitraire dans l'espace et qu'on considère l'équation de Beltrami qui lui correspond."

It was undoubtedly well to thus enunciate and emphasize the essential point in Riemann's theory, as many students of the subject, particularly those who work it up privately, are prone to regard the Riemann surface as simply an ingenious way of representing an algebraic function.

The well-known theorems of Clebsch and Lüroth are given in a very simple form in chapter XIII. In a foot-note to page 371, Picard says *à propos* of Riemann's fundamental memoir: "On trouvera dans le premier chapitre de la Thèse de M. Simart (Paris, 1882) un exposé très complet et très rigoureux des théorèmes relatifs à la connexité de ces surfaces, qui pour la plupart ne sont qu'énoncés par Riemann." The writer can cordially recommend this thesis to the student of Riemann's original memoir. Another foot-note on page 375 will be of interest to English readers: "C'est le géomètre anglais Clifford qui paraît avoir, le premier, remplacé la surface de Riemann sur le plan par une surface à p trous dans l'espace. Pour faire cette transformation, nous nous sommes servi de la méthode qu'il a employée dans un petit mémoire, d'une remarquable simplicité, consacré à cette théorie" [On the canonical form and dissection of a Riemann's surface, *Proceedings of the London Mathematical Society*, vol. VIII.]. The memoir of Clifford's is well known to all English readers, but it may be a little surprise to some to learn that Clifford was the first to replace the Riemann surface on the plane by a surface in three dimensional space having p holes.

The section on Riemann's surfaces closes with the application of Cauchy's theorems to functions of a complex variable on a Riemann's surface, and a proof is given of the theorem that *every function which is uniform over a Riemann's sur-*

face and which has over the surface no other singular points than poles is a rational function of x and y.

Chapter XIV. begins with a study of the periodicity of the Abelian integrals. The student of the theory of the Abelian integrals is well aware of their different determinations between given limits, these differences being linear functions of the periods and depending on the path followed from the lower limit of the integral to the upper limit. It is therefore scarcely worth while to do more than mention Picard's concise and elegant account of these different determinations of the integrals of the first category (that is, where the poles of the function to be integrated give rise to no logarithmic terms) and also of the integrals of the second category where *polar periods* arise from the fact of one at least of the poles of the function to be integrated giving rise to a logarithmic term.

We pass now to Abel's theorem, which Picard first gives in the form in which it was stated by Abel. He says: "Sous cette forme, le théorème paraît tout à fait élémentaire, et il n'y a peut-être pas, dans l'histoire de la Science, de proposition aussi importante obtenue à l'aide de considérations aussi simple."

Starting from the algebraic relation

$$(1) \qquad f(x, y) = 0,$$

consider a family of algebraic curves

$$(2) \qquad \lambda(x, y, a_1, a_2, \ldots a_r) = 0$$

depending on r arbitrary parameters $a_1, a_2, \ldots a_r$. We will suppose that λ contains these parameters rationally. The curves (1) and (2) have a certain number, say μ, of points in common,

$$(x_1, y_1), (x_2, y_2), \ldots (x_\mu, y_\mu),$$

which vary with the parameters a. The abscissas $x_1, x_2, \ldots x_\mu$ are roots of a certain equation of degree μ, say

$$(3) \qquad \theta(x, a_1, a_2, \ldots a_r) = 0,$$

whose coefficients are rational in the a's. If the axes do not occupy any particular position relative to the two curves, we can always admit that the corresponding value of y is given by

$$y = \psi(x, a_1, a_2, \ldots a_r),$$

where ψ is rational in $x, a_1, a_2, \ldots a_r$. This granted, consider any Abelian integral whatever,

$$\int_{(x_0, y_0)}^{(x, y)} R(x, y) dx,$$

where $R(x, y)$ is a rational function of x and y, and form the sum

$$S = \sum_{n=1}^{n=\mu} \int_{(x_0, y_0)}^{(x_n, y_n)} R(x, y) dx.$$

This sum is determinate to a sum *près* of multiples of the polar and cyclic periods of the integral, periods which are independent of the a's. The object of Abel's theorem is to determine the nature of this sum considered as a function of the parameters a. Denoting by δ a total differential with respect to these parameters, we have

$$\delta S = R(x_1, y_1)\delta x_1 + \ldots + R(x_\mu, y_\mu)\delta x_\mu.$$

Now by differentiating (3) we can calculate successively

$$\delta x_1, \delta x_2, \ldots \delta x_\mu;$$

substituting these values in δS and replacing the y's by their values ψ, we shall have for the coefficient of δa_1 a rational function of $x_1, x_2, \ldots, x_\mu$ and of the a's. Further, it will evidently be symmetrical with respect to $x_1, x_2, \ldots x_\mu$, and consequently the coefficient of δa_1 will be a rational function of $a_1, a_2, \ldots a_r$, and the same is of course true of the other coefficients. We have therefore

$$\delta S = P_1(a_1, a_2, \ldots a_r)\delta a_1 + \ldots + P_r(a_1, a_2, \ldots a_r)\delta a_r,$$

where the P's are rational functions of $a_1, a_2, \ldots a_r$.

This equality constitutes Abel's theorem in its primordial form; it expresses the fact that S is an algebraico-logarithmic function of the parameters a. In fact, integration of the total differential in the second member leads necessarily to an expression of the form

$$\phi + \Sigma A \log \Phi,$$

the A's denoting constants and ϕ and Φ rational functions of the a's.

Next follows the well-known form of Abel's therorem when applied to Abelian integrals of the first kind. The derivation of this most important form of the theorem is so brief and simple that it may be given here.

We consider the Abelian integrals of the first kind,

$$\int_{(x_0,\, y_0)}^{(x,\, y)} R(x,\, y)dx,$$

that is the integrals which remain finite for every value of x and y on the Riemann's surface. Apply Abel's theorem to such an integral. In the first place, the algebraico-logarithmic function of the a's must remain finite for every finite or infinite value of the a's, since S itself remains always finite. Now a function of the form

$$\phi + \Sigma A \log \Phi,$$

where ϕ and Φ represent rational functions of the a's and where the A's are constants, cannot remain finite for every value of the parameters a; it must therefore reduce to a constant It follows then that the sum S is independent of the parameters a. For integrals of the first kind we can therefore enunciate Abel's theorem as follows: *The sum*

$$\sum_{n=1}^{n=\mu} \int_{(x_0,\, y_0)}^{(x_n,\, y_n)} R(x,\, y)dx,$$

where $(x_n,\, y_n)$ *denote the points of intersection of the curves* (1) *and* (2), *points which vary with the parameters* **a**, *does not depend on these parameters.*

The sum has a constant value, leaving aside, of course, linear functions of certain fixed periods which can always be introduced by varying the path between $(x_0,\, y_0)$ and $(x_n,\, y_n)$. The theorem for this case can also be put in the form

$$R(x_1,\, y_1)dx_1 + R(x_2,\, y_2)dx_2 + \ldots + R(x_\mu,\, y_\mu)dx_\mu = 0,$$

the d's denoting total differentials with respect to $a_1, a_2, \ldots, a_r$.

The great importance of this special form of Abel's theorem in analysis and in the theory of algebraic curves seems to the writer a sufficient justification for reproducing here Picard's very concise and simple statement of it.

The consideration of integrals of the first kind is now taken up and it is first shown that these integrals are necessarily of the form

$$\int \frac{Q(x,\, y)dx}{f_y'},$$

where $Q(x,\, y)$ is a polynomial, and then the necessary and sufficient condition is found in order that an integral of this form may really be an integral of the first kind; viz., that $Q(x, y)$ must be a polynomial of degree $m - 3$ in x and y and the curve

$$Q(x, y) = 0$$

must have as multiple points of order $i - 1$ the multiple points of order i of the curve $f(x, y) = 0$.

An exceedingly interesting investigation now follows concerning the number of linearly independent integrals of the first kind, but it cannot be entered into here. Certain fundamental theorems concerning integrals of the first kind are now given, and incidentally a new demonstrution is given of the theorem concerning the number of linearly independent integrals of the first kind. In finishing the subject of integrals of the first kind it must suffice to merely state the following theorem established on page 409: *We can form an integral of the first kind for which the real parts of the 2p periods have arbitrary given values.* This chapter closes with a brief but most admirable account of the integrals of the second and third kinds.

Chapter XV. of vol. II. is entitled *Des fonctions uniformes sur une surface de Riemann.* It is impossible to give here any adequate account of this most interesting and important chapter; it will not submit to condensation—any condensation would merely be mutilation. The different sections of the chapter are as follows : I. Decomposition of rational functions of x and y into simple elements. II. The Riemann-Roch theorem. Special functions. This section contains Brill and Nöther's law of reciprocity. III. Birational transformations of curves into themselves. This section contains a demonstration by Picard of an important theorem of Schwarz, viz., *the curves of genus zero and of genus one are the only ones which can be transformed into themselves by a birational substitution involving an arbitrary parameter.* This demonstration is peculiarly interesting, as Picard obtains it by following the same path which led him to an analogous theorem for algebraic surfaces in his celebrated "mémoire couronné" of 1888: "*Mémoire sur la théorie des fonctions algébriques de deux variables indépendantes,*" chap. III. (*Journal de Mathématiques,* 1889). IV. Classes of algebraic curves. Normal curves. V. Curves of genus two. In section IV. there are so many interesting theorems that it is difficult to select any one or two as illustrations of the section. From section V., however, we may quote one theorem which is established, viz., *every curve of genus two corresponds point to point to a curve of the fourth order having one double point.*

Chapter XVI. is entitled *Théorèmes généraux relatifs à l'existence des fonctions sur une surface de Riemann.*

I. Statement of the question; preliminary theorems. II. Existence of harmonic functions on an open Riemann surface. III. Existence of harmonic functions on a closed Riemann surface. IV. Functions of a complex variable on a Riemann surface. In this section a demonstration of the fundamental theorem of the chapter is given, viz., *to an arbitrarily given Riemann surface corresponds a class of algebraic curves.* The importance of this theorem, as placing in its true light the conception of the Riemann surface, can hardly be exaggerated. V. Moduli of a class of algebraic curves. This is a most important section, but no account of it can be given here. VI. Existence theorems for Beltrami's equation corresponding to any surface whatever. This section deals with Beltrami's generalization of Laplace's equation to any surface whatever. This generalization will be referred to later. In the present connection it will be sufficient to quote the fundamental theorem concerning this generalization: *To the given surface S in space having p holes corresponds uniformly an algebraic curve of genus p.* The footnote to page 493 gives an interesting history of this theorem. A limiting case of the surface considered gives the following theorem due to Schottky. To every plane disk with p holes there corresponds a class of algebraic curves.

Chapter XVII., which closes vol. II., has for title *Courbes des genres zero et un.* It is divided into three sections: I. Unicursal curves. II. Curves of genus one. III. Generalities on doubly periodic functions. The theorems in this short chapter are well known and need not be recapitulated. It is sufficient to say that they are presented in the graceful way peculiar to Picard. An historical remark in a footnote to page 498 is interesting. Picard says: "Les mots *courbes unicursales* ont été employés pour la première fois par M. Cayley."*

We return now to vol. I. Chapters III., IV., and V. are respectively entitled *Intégrales curvilignes, Des intégrales doubles,* and *Des Intégrales multiples.* These chapters, interesting and important as they are, must be passed over with a mere mention; they will be of great interest to the physicist as well as to the mathematician. The conditions that a line integral shall depend only on the limits and not on the path and that a surface integral shall depend only on the limiting contour of the surface are given in an exceedingly elegant form; the fundamental notion of the calculus of variations is introduced here. The sections dealing with the question of the roots common to two equations (chap. III.), and the roots common to three equations (chap. IV.) are of particular interest, as Picard has himself probably said the last word on the subject of the number of roots common to n simultaneous equations in his memoir in the *Journal de Mathématiques* (1892). The reader may recall a brief discussion between Picard and Kronecker, just before the latter's death, on this subject. It is not necessary to enter into that discussion here; it will suffice to say that Kronecker gave a formula for the *difference* between the number of roots contained in a given region for which a certain determinant is positive and those for which it is negative. The *exact* number of roots in the region is thus not given by Kronecker's formula. Picard has, however, shown how the difficulty which presents itself in Kronecker's formula can be overcome, and proves that the number of roots common to two equations and contained in a certain contour can be represented by a double integral; and, as already mentioned, he gives the analogous theorem for the case of n simultaneous equations. These latter results are contained in chapter VII. of vol. II. The final theorem of this chapter may be quoted: "*On peut donc par suite trouver, par un calcul algébrique régulier, le nombre des racines des deux équations,*

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* See *Comptes rendus*, vol. LXII.

$$f(x, y) = 0,$$

$$\phi(x, y) = 0,$$

(f et φ étant deux polynômes) contenues dans un contour défini par les n inégalités,

$$\lambda_i(x, y) < 0, \qquad (i = 1, 2, \ldots n),$$

les λ étant des polynômes."

Part second of volume I. treats of Laplace's equation and its applications and of developments in series. This second part is also of much interest to physicists, dealing as it does with Laplace's equation, Dirichlet's principle, the theory of attraction and of the potential, and particularly with trigonometric series. These subjects are all classical and need not be further spoken of, though it may be mentioned that the theorems of Cantor and Schwarz probably appear here for the first time in any treatise on analysis.

In taking up the subject of multiple series Picard first gives a generalization of Cauchy's rule for convergence in the case of simple series with positive terms. This rule is readily applied to series whose general term is of the form

$$\frac{1}{[f(m_1, m_2, \ldots m_p)]^a},$$

f denoting a definite and positive quadratic form. An interesting example of a double series is given by the absolutely and uniformly convergent development in a double trigonometric series of a function $f(x, y)$, which, together with its first four partial derivatives, is continuous and possesses the property defined by the equations

$$f(x+2\pi, y) = f(x, y), \qquad f(x, y+2\pi) = f(x, y).$$

Another illustration of double series is taken from the theory of the elliptic functions, and then the author considers a quadruply periodic function of two variables investigated by himself in a paper contained in the *Bulletin de la Société Mathématique*, 1889. This function is the sum of the double series

$$\sum_{m=-\infty}^{m=+\infty}\sum_{n=-\infty}^{n=+\infty}\frac{e^{x+m\alpha+n\beta}}{(1+e^{x+m\alpha+n\beta})^2}\cdot\frac{e^{y+m\alpha'+n\beta'}}{(1+e^{y+m\alpha'+n\beta'})^2},$$

where α, β, α', β' are real numbers such that the determinant $\alpha\beta' - \alpha'\beta$ is different from zero. This series is by very simple transformations changed into a trigonometric series.

The last section of this chapter is of special interest to the student of the more recent developments in the theory of functions. It treats of series where the indices are not arbitrary, that is, do not make up all possible systems of integers, but are confined to the substitutions of a certain group. Picard then shows how it is possible to form functions of two variables which are entirely analogous to Poincaré's thetafuchsian functions. These hyperfuchsian functions are not studied here, but it is probable that the author will return to them in another volume.

Part III. of vol. I. is devoted to geometrical applications of the infinitesimal calculus. The subjects first treated are the theories of envelopes, developable surfaces, ruled surfaces, with a special study of the most general ruled surface of the third order, congruences, and complexes of lines. The essential properties of the focal surface of a congruence are investigated and a proof is given of Dupin's theorem that *the necessary and sufficient condition that the straight lines of a congruence shall be normal to a surface is that the tangent planes to the two nappes of the focal surface corresponding to any generatrix whatever shall be rectangular.*

Here follow some brief considerations of complexes, especially the linear complex, and chapter XI. closes with the theorem that *the tangents of a skew cubic belong to a linear complex.*

In chapter XII. the theory of contact of plane and skew curves is discussed, and among other things are derived Cayley's formulæ for skew algebraic curves, which are analogous to Plücker's formulæ for plane algebraic curves. In this chapter is also contained an investigation of curves whose tangents belong to a linear complex.

Chapter XIII. deals with the curvature and torsion of skew curves, and chapter XIV. with curves traced on surfaces. Here the theorems of Euler and Meunier are first obtained, and then the subject of lines of curvature is taken up and the theorems of Joachimsthal and Dupin are established. Surfaces which are the envelopes of spheres, and in particular Dupin's cyclide, generalities on asymptotic lines and asymptotic lines on certain ruled surfaces form the subjects of the closing sections of this rather long and very interesting chapter. Chapter XV. is entitled *Surfaces applicables. Representation conforme. Cartes géographiques.* The expression for the square of the element of arc of a curve on a surface is first found, and the subject of surfaces applicable to one another, and in particular to the plane, is developed. In the second section the conform representation of a plane upon a plane is studied, and the third section gives some examples of conform representation and a little introduction to the linear substitutions

$$\left(z, \frac{az+b}{cz+d}\right),$$

for which $ad - bc = 1$.

After obtaining some of the familiar properties of such a substitution he defines a *group* and then a *discontinuous group*, viz. Poincaré's Fuchsian group. Following this he considers with more detail the group formed by the substitutions

$$\left(z, \frac{az+b}{cz+d}\right),$$

where a, b, c, d are four real integers, satisfying the relation

$$ad - bc = 1.$$

This group is first proved to be discontinuous, and then it is shown that it leads to a division of the half-plane into an infinite number of triangles. The proof which Picard gives of this result is based upon the arithmetical theory of the reduction of definite quadratic forms, which makes it necessary to give a very brief account of these forms—or, more exactly, of the notion of a reduced quadratic form (*forme quadratique réduite*). Picard then shows very briefly how the notion of connecting the theory of the substitution

$$\left(z, \frac{az+b}{cz+d}\right)$$

with the theory of quadratic forms can be extended to obtain the substitutions for a half-space analogous to those found for the half-plane. This investigation is contained in a paper by Picard in the *Bulletin de la Société Mathématique* for 1884. The title is *Sur un groupe de transformations des points de l'espace situés du même côté d'un plan.* The final section of this chapter and volume is a very brief one on map projections. It is a matter of regret to the writer that this beautiful part III. of vol. I. has to be noticed so briefly, but he has thought it better to touch lightly on these applications of analysis to geometry in order to leave more space for the analysis itself.

The closing chapters of vol. II. have already been noticed and need not be particularly referred to again. The principal part of vol. II. is devoted to harmonic and analytical functions. Picard says in the introduction: "Sans négliger le point de vue de Cauchy dans la théorie de ces dernières fonctions, je me suis surtout attaché à une étude approfondie des fonctions harmoniques, c'est-à-dire de l'équation de Laplace; une grande partie de ce volume est consacrée à cette équation célèbre, dont dépend toute la théorie des fonctions analytiques. Je me suis arrêté longuement sur le principe de Dirichlet, qui joue un si grand rôle dans les travaux de Riemann, et qui est aussi important pour la physique mathématique que pour l'analyse."

The first chapter begins with the definition of a function of a complex variable and the familiar conditions of Cauchy that it shall be monogenic—that is, if the function be denoted by $u + iv$ we must have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Following this is given the interesting generalization due to Beltrami of these equations to the case of any surface Σ on which the element of length is given by the equation

$$ds^2 = Edp^2 + 2Fdpdq + Gdq^2.$$

Beltrami's generalized equations are

$$\frac{\partial v}{\partial p} = \frac{F\dfrac{\partial u}{\partial p} - E\dfrac{\partial u}{\partial q}}{\sqrt{EG - F^2}};$$

$$\frac{\partial v}{\partial q} = \frac{G\dfrac{\partial u}{\partial p} - F\dfrac{\partial u}{\partial q}}{\sqrt{EG - F^2}}.$$

The combination $u + iv$ can now be called a complex function of the point (p, q) on the surface Σ. There follows now a careful study of Laplace's equation and then a most important extension to the general linear partial differential equation of the second order, with two independent variables, of some of the results obtained for Laplace's equation. The author's important contributions to the theory of these equations are well known to all students of differential equations. The last section of chapter I. is principally concerned with Neumann's method for the solution of Dirichlet's problem.

Chapter II. is a very important chapter in the theory of functions: it is entitled *Développments en series et prolongement analytique des fonctions harmoniques et des fonctions d'une variable complexe.*

In concluding certain generalities concerning harmonic functions and developments in series the author gives a proof of the following theorem of Harnack's: *Let there be given a series*

$$u_0 + u_1 + u_2 + \ldots + u_n + \ldots$$

of harmonic functions which are all positive inside an area limited by a contour C. If this series is convergent at a point O in the interior of the area, it will be convergent at every point in the interior of the area and will represent a harmonic function.

The pages devoted to the subject of the extension of an analytical function, to the examples of functions which cannot be extended (Freedholm's, for example), and to the recent theorems given by Hadamard, particularly his theorem relative to the region of convergence of a power series (Taylor's series), are full of most interesting and valuable results, but can only be alluded to here. The next two chapters, III. and IV., are devoted wholly to Dirichlet's problem, and contain an account of the methods of Schwarz and Poincaré—the latter method being developed in a memoir contained in vol. IX. of the *American Journal of Mathematics.*

Chapter V. contains a direct study of functions of a complex variable and begins by establishing certain well-known and general theorems of Cauchy. The second section, dealing with poles and essential singularities, gives further theorems of Cauchy's and introduces Weierstrass's notion of essential singular points and a mere mention of a most important theorem of the author's on entire functions. Section III. contains familiar elementary examples of functions of a complex variable. Section IV. deals very briefly with convergent products, and section V. is concerned with the decomposition of uniform functions into primary factors. A very elegant and rigorous demonstration of Weierstrass's results is given first and then a most interesting generalization is given of these results (which are too well known to need statement).

Picard shows that a formula similar to that of Weierstrass can be found for uniform functions which are continuous for all points of the plane except such as are situated upon a circumference C of radius R and having the origin as centre.

Let $A_1, A_2, \ldots A_n, \ldots$ be a series of quantities such that on writing $A_n = \rho_n e^{i\alpha_n}$ we have

$$| \rho_n - R | \geqq | \rho_{n+1} - R |$$

and further $\lim\limits_{n=\infty} \rho_n = R$. It is now shown that we can form an expression depending on z which shall be uniform and continuous in all points of the plane, points of the circumference excepted, which shall represent an analytical function of z inside and outside the circumference and shall vanish for the values $A_1, A_2, \ldots A_n, \ldots$ of z.

Another series of quantities

$$B_1, B_2, \ldots B_n, \ldots$$

are taken on the circumference C and such that

$$\lim_{n=\infty} (A_n - B_n) = 0.$$

The product

$$\prod_{n=1}^{n=\infty} \frac{z - A_n}{z - B_n} e^{\phi_n(z)},$$

where

$$\phi_n(z) = \frac{A_n - B_n}{z - B_n} + \frac{1}{2}\left(\frac{A_n - A_n}{z - B_n}\right)^2 + \ldots + \frac{1}{n-1}\left(\frac{A_n - B_n}{z - B_n}\right)^{n-1},$$

is shown to be convergent and to represent an analytical function $G(z)$ having the properties mentioned. Two interesting examples of this theorem are given. The chapter closes with a theorem due to Painlevé and which is derived from the fundamental Cauchy formula

$$F(x) = \frac{1}{2\pi i} \int_C \frac{F(z)}{z - x} dz.$$

The theorem is: *Every function which is holomorphic in an area limited by a convex contour can be developed in this area in a series of polynomials.*

It must suffice simply to mention the matters treated in chapter VI., which is devoted to applications of Cauchy's theorems: Investigations of certain definite integrals; Developments in series of rational fractions; (here some important remarks are made concerning the Weierstrass decomposition of a function into primary factors;) Cauchy's method for obtaining Fourier's and analogous series; Number of roots of an equation contained in a contour; and the Theory of indices.

Chapter VII., on the number of roots common to two simultaneous equations, has already been mentioned.

Chapter VIII. has for title *Intégrales des fonctions non-uniformes.* The first section shows the different determinations which an hyperelliptic integral of the type $\int_{x_0}^{x}\frac{f(x)dx}{\sqrt{R(x)}}$ ($f(x)$ a polynomial) can have when the path between the limits is varied. The second section considers the integrals of the first kind and the reduction of the number of periods; here a proof is given that there must be at least two distinct periods. The elementary properties of the periods of the elliptic integral of the first kind are given now, including a proof of the theorem that the ratio of the two periods is imaginary. Section III. contains an example of a non-uniform function represented by integrals, an application to the hypergeometric series, and finally some important properties of the ratio of the periods of an elliptic integral regarded as a function of the modulus. The employment of this function now enables the author to give proofs of his two celebrated theorems on uniform functions. The first is: *An entire function $G(z)$ which can never become equal to two values a and b is necessarily a constant.* For the second theorem we consider a uniform function $f(z)$ having throughout the plane only poles as singular points; Picard then shows that *there cannot be more than two finite values a and b which this function cannot take for a finite value of the variable; if there are more than two, the function reduces to a constant.*

Chapter IX. is a most admirable introduction to the study of functions of several independent variables. This chapter is rendered particularly interesting and valuable by containing the author's presentation of Poincaré's extension of Cauchy's fundamental theorem for one complex variable to the case of two such variables. As it is impossible to give any adequate account of this beautiful piece of analysis here it seems better to pass it over with a mere mention. The chapter closes with a study of Lagrange's formula for one and two equations.

Chapter X. resumes the theory of conform representation, the elementary propositions in which were given in vol. I. In this chapter the author is particularly occupied with the question of the representation of a given area upon another area equally given. It is necessary first to give a definition of an *arc of an analytic curve.* Suppose a curve such that the coordinates x and y of an arbitrary point are analytical functions of a parameter t,

$$x = f(t), \qquad y = \phi(t),$$

f and ϕ being supposed holomorphic functions of t in the region of the real value $t = t_0$, the coefficients of the developments in series according to powers of $t - t_0$ being of course real. We say, then, that this arc is analytic; furthermore, this analytic arc will be *regular* at the point corresponding to $t = t_0$ if we can choose the parameter t on which x and y depend analytically in such a way that $f'(t_0)$ and $\phi'(t_0)$ are not both zero at the same time. A determinate arc $\alpha\beta$ is said to be regular if it is regular at all its points. Suppose we have a closed contour C, and let us admit that a portion $\alpha\beta$ of this contour is a regular arc of an analytical line. Assign now a succession of values along the contour, and suppose that the ensemble of values along the arc $\alpha\beta$ forms an analytical function of the parameter t. Under these conditions the following theorem due to Schwarz is demonstrated: The harmonic function taking the given values along the contour can be prolonged analytically beyond the arc $\alpha\beta$.

The conform representation of a simple area on a circle is next taken up, and then a presentation is given of Schwarz's method for Dirichlet's principle. The chapter concludes with the consideration of a simple case of two areas limited by several contours, and it is shown how a function which is holomorphic in the interior of an ellipse can be developed in a series of polynomials.

Chapter XI. contains general theorems in differential equations; the theorems are classical, but their mode of presentation is modern, and it need hardly be said that it is both elegant and rigorous. The theorems of the existence of an integral for a differential equation, or of a system of integrals for a system of ordinary or of partial differential equations, occupy the entire chapter. Cauchy's methods are given, and also the author's own method by successive approximations, which is familiar to the readers of this BULLETIN by Dr. Fiske's translation. The *unique* determination of a system of integrals for given initial values is emphasized strongly by the author for reasons which need not be gone into.* The chapter is a thoroughly satisfactory one on this fundamental question in the theory of differential equations.

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* See page 317.

Chapter XII. contains some applications of the general theorems, and begins with some of the well-known theorems of Briot and Bouquet and then follows a most important theorem due to Painlevé and the latter's notion of fixed and movable critical points. The writer has given a brief explanation of what Painlevé means by these terms in another number of this BULLETIN, and it need not be repeated here. Riccati's equation is next studied, and the chapter closes with an account of the inversion of the elliptic integral and of certain entire functions associated with the elliptic functions. The fuller study of the subject of differential equations and of the functions defined by them is reserved for another volume.

The writer is quite conscious of the inadequacy of the preceding notice to give a satisfactory idea of this most important work of M. Picard's. The attempt has been made to show how in each theory or its application M. Picard goes at once to what is essential and in particular in the applications how he has selected really important problems in analysis, geometry, and mathematical physics. No applications are given simply because they afford pretty exercises in analysis or give rise to very symmetrical sets of formulæ. It is customary to say something about the typography of a book reviewed and concerning errata. As for the former, it is hardly necessary to comment on Gauthier-Villars' manner of getting up a book. As for the latter, they are too few and trifling to mention; still one might mention one which the reader will not find out is an error until he has read nearly a page further. In the third line from the bottom of page 45, vol. I., we find the words: "Je me place d'abord dans le premier cas." Instead of the "first case" it should be the "*second case.*"

T. CRAIG.

BALTIMORE, Oct. 10, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, October 7, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Mr. John M. Colaw, Monterey, Va.; Mr. David Lyman Pettegrew, Worcester, Mass.; Dr. Isaac J. Schwatt, University of Pennsylvania; Professor David Eugene Smith, Michigan State

Normal School. Professor E. Study of Marburg was present and was introduced to the Society by the president. Being asked to address the Society, he first communicated a brief note on monogenic functions by Professor T. Craig (see the next number of the BULLETIN). He then made some remarks concerning researches made by himself jointly with Professor F. Engel of Leipzig in the theory of groups, and produced an example of a continuous group in which the finite transformations were not all generated by infinitesimal transformations (see Chapman's report on Lie's theory, BULLETIN, vol. II., No. 4, p. 69). This group is the so-called special linear homogeneous group:

$$x' = \alpha x + \beta y, \qquad y' = \gamma x + \delta y, \qquad \alpha\delta - \beta\gamma = 1.$$

Excepting the identical transformation $x' = x$, $y' = y$, and the involutory transformation $x' = -x$, $y' = -y$, we may say that in this group every given finite transformation is contained in one single one-branch group (Chapman, p. 66), which is, in general, generated by an infinitesimal transformation. But this group may break up into two parts, or "hosts" (Chapman, p. 62). The transformations of the first host, containing the identical transformation, once more form a continuous group; the transformations of the second host, containing the said involutory transformation, do not form a group and cannot be generated by any infinitesimal transformation whatever belonging to the special linear homogeneous group. These considerations and others associated with them will be published *in extenso* in the *Berichte* of the Saxon Academy of Sciences at Leipzig.

A paper by Professor W. H. Echols entitled "Note on the theory of functions" was read by the secretary. In this paper the development of functions of a complex variable in series of factorial functions was treated in a manner somewhat analogous to that usually employed in the derivation of Taylor's and Laurent's theorems.

THE Annual Meeting of the NEW YORK MATHEMATICAL SOCIETY will be held on Thursday, December 28, at *half-past three* in the afternoon instead of at *half-past four* as heretofore announced. Professor Simon Newcomb of Johns Hopkins University has consented to deliver an address to the Society at this meeting, and has chosen as his topic "Modern Mathematical Thought."

THE Mathematical Club of the University of Virginia has recently been reorganized. It is to meet weekly during the session on Mondays at 3:30 P.M. Dr. James M. Page is president and Mr. Edgar Odell Lovett is secretary. At the first

eting Professor Echols gave a contribution from his inves-
ations in the theory of functions of a real variable. Original
ers are expected also from Dr. Page on his researches in
's theory of infinitesimal transformations, and from other
mbers of the club. T. S. F.

DECIDED novelty in mathematical journalism is *L'inter-
diare des mathématiciens,* of which the first number is an-
inced to appear in January 1894. It will be the principal
ect of this new monthly to furnish, in answer to inquiries,
ormation on mathematical subjects, solutions of problems,
liographical references, etc. Both questions and answers
l be published; but anything not directly pertaining to the
estions proposed, such as memoirs or papers on special sub-
ts, is rigorously excluded.

3y publishing the question (either with the name of the
poser, or only with his initials, or under a pseudonym if
ferred) it is desired to engage the active co-operation of the
ders of the journal and to establish communication between
sons who can be helpful to each other. To quote from the
liminary announcement of the editors: "In most of the
sting mathematical journals questions are proposed; but
ierally the proposer himself has a solution. In our journal
te a different idea will be carried out. Problems and ques-
ns are to be offered just for the reason that the proposer
hes to obtain a solution or at least indications concerning it.
netimes, also, the object may be to obtain rapidly a result
t could otherwise be found only by long and tedious work.
[n the 17th century men of science addressed challenges to
h other and tried to conceal their methods of solution;
ence has gained largely by this emulation. But in our age
conditions are changed: science is spread widely; every
iolar desires to make known his discoveries as soon as pos-
le; and a sort of collective effort is being substituted with
at advantage for the individual work of our ancestors. It
this co-operation among mathematicians that we wish to
ter and farther develop by economizing the time spent use-
sly in researches that were made before by others."

[n view of the extreme specialization of modern mathematics
l of the consequent difficulty of obtaining the best informa-
n on particular questions, the help that might be afforded
a journal of this kind to a worker in any special branch can
rdly be overestimated. The names of the editors, MM. C.-A.
isant and Émile Lemoine, are a sufficient guarantee of the
e and efficient execution of the plan. Gauthier-Villars is
publisher; the price of subscription, within the limits of
postal union, is 6 francs annually.

Suitable questions are solicited for the first number; the proof-sheets of the questions received for this number will be sent to the correspondents so as to obtain their collaboration from the very beginning. Communications are to be addressed to the firm of Gauthier-Villars, 55, quai des Grands-Augustins, Paris; or to either of the editors: C.-A. Laisant, 162, avenue Victor-Hugo, and E. Lemoine, 5, rue Littré, Paris. A. Z.

Among the recent announcements of the Cambridge University Press we notice "The scientific papers of John Couch Adams," vol. I., edited by Dr. W. G. Adams, F.R.S., with a memoir by J. W. L. Glaisher, and "A treatise on spherical astronomy" by Sir Robert S. Ball.

Dr. Charles H. Chapman has accepted the presidency of the University of Oregon, at Eugene, Oregon. T. S. F.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Brewster (D.). Life of Sir Isaac Newton. New ed. (Castle Series.) London, Gall, 1893. 8vo. 340 pp. 2s.

Carrara. Saggio d'introduzione alla teoria delle quantità complesse geometricamente rappresentate. Cremona, 1893.

Chittenden (J. B.). A presentation of the theory of Hermite's form of Lamé's equation, with a determination of the explicit forms in terms of the $\wp$ function for the case $n = 3$. [Diss.] Leipzig, (Teubner,) 1893. 8vo. 85 pp. Mk. 2.80

Delin (C.). Ueber zwei ebene Punktsysteme, die algebraisch auf einander bezogen sind. [Diss.] Lund, Möller, 1893. 8vo. 85 pp. 1 plate. Mk. 1.60

Fegerl. Ueber die unendlich oft iterierten Funktionen. [Progr.] Ostrau, 1893.

Frege (G.). Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet. Vol. I. Jena, Pohle, 1893. 8vo. 32 and 253 pp. Mk. 12.00

Gravelius (H.). Lehrbuch der höheren Analysis. Zum Gebrauche bei Vorlesungen an Universitäten und technischen Hochschulen. Vol. I: Lehrbuch der Differentialrechnung. Berlin, Dümmler, 1893. 8vo. 8 and 324 pp. Mk. 6.00

Gschnitzer (F.). Die Affinität in der Ebene nebst ihren speciellen Fällen, der Ähnlichkeit, Congruenz und Flächengleichheit, in analytischer Behandlung. [Progr.] Brixen, 1893. 8vo. 27 pp. Illustrated.

Huth (M.). Ueber Kurven von konstanter Steigung auf gegebenen Flächen. [Progr. No. 578.] Stollberg, 1893. 4to. 20 pp.

Jahresbericht der Deutschen Mathematiker-Vereinigung. Herausgegeben im Auftrage des Vorstandes von G. Cantor, W. Dyck, E. Lampe. Vol. II: 1892–3. Berlin, Reimer, 1893. 8vo. 4 and 156 pp. 2 plates. Mk. 4.50

Obenrauch (F. J.). Monge, der Begründer der darstellenden Geometrie als Wissenschaft. Eine mathematisch-historische Studie. [Progr.] Brünn, 1893. 8vo. 33 pp. Mk. 2.00

Peano (G.). Lezioni di analisi infinitesimale. 2 vols. Torino, Bocca, 1893. 8vo. 652 pp. Fr. 15.00

Preat y Lluch (P.). Problemas gráficos de estereometria. Madrid, 1893. 8vo. 89 pp. Mk. 3 50

Thompson (H. D.). Hyperelliptische Schnittsysteme und Zusammenordnung der algebraischen und transcendentalen Thetacharacteristiken. [Diss.] Göttingen, 1893. 4to. 33 pp. Illustrated.

Verzeichnis der seit 1850 an den deutschen Universitäten erschienenen Doctor-Dissertationen und Habilitationsschriften aus der reinen und angewandten Mathematik. Herausgegeben auf Grund des für die Universitäts-Ausstellung in Chicago erschienenen Verzeichnisses. München, Dyck, 1892. Mk. 1.70

II. ELEMENTARY MATHEMATICS.

Adam (W.). Geometrische Analysis und Synthesis. Eine Sammlung von 636 planimetrischen Konstruktions-Aufgaben mit reingeometrischer Lösung. Für höhere Lehranstalten, sowie zum Gebrauch beim Selbstunterricht systematisch geordnet und bearbeitet. 2d ed. Potsdam, Stein, 1893. 8vo. 4 and 292 pp. Mk. 4.00

Bardey (E.). Algebraische Gleichungen nebst den Resultaten und den Methoden zu ihrer Auflösung. 4th ed. Leipzig, Teubner, 1893. 8vo. 12 and 378 pp. Mk. 6.00

Barthels (E. L.). Lehrbuch der Stereometrie und Trigonometrie in ausführlicher Darstellung. Nebst einem Anhange, enthaltend: die Regeln über Potenz-, Wurzel-, Gleichungs-, Reihen- und Logarithmenlehre. Wiesbaden, Sadowsky, 1893. 8vo. 8 and 120 pp. Illustrated. Mk. 2.80

Briggs (W.) and **Edmonson** (J. W.). The geometrical properties of the sphere. (Univ. Corr. Coll. Tutorial Series.) London, Clive, 1893. 8vo. 1s. 6d.

Briot (C.). Leçons d'algèbre. 16me éd., rev. par E. Lacour. Part II. Paris, 1893. 8vo.

Eggers (W.). Lehrbuch der darstellenden Geometrie. Part I: Die Elemente der darstellenden Geometrie. Leipzig, Seemann, 1893. 8vo. 8 and 52 pp. 5 plates. Boards. Mk. 1.50

Emmerich (A.). Der Koordinatenbegriff und einige Grundlehren von den Kegelschnitten. Für den Schulunterricht bearbeitet. Essen, Baedeker, 1893. 8vo. 4 and 34 pp. 5 plates. Mk. 0.80

Genau (A.). Die Logarithmen und die ebene Trigonometrie. Büren, Hagen, 1893. 8vo. 47 pp. Illustrated. Boards. Mk. 1.10

Handel. Elementar-synthetische Kegelschittslehre. Berlin, Weidmann, 1893.

Hauck. Lehrbuch der Stereometrie. Auf Grund von Kommerells Lehrbuch neu bearbeitet. 7th ed. Tübingen, Laupp, 1893. 8vo. 226 pp. Mk. 2.40

Heller (J. F.). Methodisch geordnete Sammlung von Aufgaben aus der darstellenden Geometrie für Realschulen. Part III. Wien, Hölder, 1893 8vo. 49 pp. Mk. 0.80

Hercher (B.). Lehrbuch der Geometrie zum Gebrauche an Gymnasien, nach den neuen preussischen Lehrplänen bearbeitet. (3 parts.) Leipzig, Jacobsen, 1893. 8vo. 12 and 182 pp. Mk. 3.00

Hertzer (H.). Fünfstellige Logarithmentafeln. 3d ed. Berlin, Gaertner, 1893. 8vo. 4 and 88 pp. Boards. Mk. 1.20

Lackemann (B.). Die Elemente der Geometrie. Theil I: Planimetrie. 3d ed. Breslau, Hirt, 1893. 8vo. 108 pp. Illustrated. Boards. Mk. 1.25

Lengauer (J.) Die Grundlehren der ebenen Geometrie. Ein Leitfaden für den Unterricht, mit Uebungsaufgaben. 4te Auflage der Geometrie von A. Stegmann. Kempten, Kösel, 1893. 8vo. 8 and 180 pp. Illustrated. Mk. 2.00

Matthiessen (L.). Uebungsbuch für den Unterricht in der Arithmetik und Algebra. Nach der Aufgabensammlung von Heis bearbeitet. 3d ed. Köln, Du Mont-Schauberg, 1893. 8vo. 8 and 254 pp. Mk. 2.00

Mink (W.). Lehrbuch der Geometrie als Leitfaden beim Unterrichte an höheren Lehranstalten. Theil I: Planimetrie. 9te Auflage, bearbeitet von E. Arndt. Berlin, Wiegandt und Schotte, 1893. 8vo. 6 and 110 pp. Illustrated. Mk. 1.50

Močnik (F. von). Geometrische Formenlehre. 2d ed. Wien, Gerold, 1893. 8vo. 4 and 123 pp. Illustrated. Mk. 1.40

——— Lehrbuch der Geometrie. 3d ed. Wien, Gerold, 1893. 8vo. 178 pp. Illustrated. Cloth. Mk. 2.00

——— Die geometrische Formenlehre in der Volksschule. Eine Anleitung für Lehrer zur Ertheilung des geometrischen Unterrichtes. 4th ed. Leipzig, Freytag, 1893. 8vo. 92 pp. Illustrated. Boards. Mk. 1.10

Ribi (D.). Aufgaben über die Elemente der Algebra, methodisch geordnet und in engem Anschluss an den Leitfaden von M. Zwicky. Herausgegeben von M. Zwicky. 3tes Heft. 6th ed. Bern, Schmid, 1893. 8vo. 32 pp. Mk. 0. 50

Roeder (H.). Lehrsätze und Aufgaben aus der Planimetrie. 2d ed. Breslau, Hirt, 1893. 8vo. 96 pp. Boards. Mk. 1.00

——— Aufgaben aus der ebenen Trigonometrie. Auflösungen. Breslau, Hirt, 1893. 8vo. 62 pp. Mk. 1.25

Rumper und **Blind**. Lehrbuch der Geometrie. Theil I: Planimetrie. 2d ed. Theil II: Trigonometrie und Stereometrie. Leipzig, Ahn, 1893.

Schlotke (J.). Lehrbuch der darstellenden Geometrie. Theil II: Schatten- und Beleuchtungslehre. Dresden, Kühtmann, 1893. 8vo. 4 and 60 pp. Illustrated. Mk. 2.00

Schwering (K.). Anfangsgründe der Arithmetik und Algebra für höhere Lehranstalten, nach den neuen Lehrplänen bearbeitet. Freiburg, Herder, 1893. 8vo. 8 and 80 pp. Cloth. Mk. 1.30

Schwering (K.). Trigonometrie für höhere Lehranstalten, nach den neuen Lehrplänen bearbeitet. Freiburg, Herder, 1893. 8vo. 8 and 52 pp. Illustrated. Cloth. Mk. 1.10

Streissler (J.). Die geometrische Formenlehre in Verbindung mit der Anschauungslehre. 8th ed. Triest, Schimpff, 1893. 8vo. 44 pp. 2 plates. Mk. 1.20

Töpfer (H.). Lehrbuch der Planimetrie. Sondershausen, Eupel, 1893. 8vo. 4 and 188 pp. Mk. 2.20

Vetters (K.). Abriss der darstellenden Geometrie. Theil I: Orthogonalprojektivische Darstellung von Punkten, Geraden, Ebenen und ebenflächigen Körpern. Chemnitz, Focke, 1893. 8vo. 4 and 84 pp. 15 plates. Mk. 3.00

Wagner (H.). Lehrbuch der ebenen Geometrie und Aufgabensammlung für Realschulen. 2d ed. Hamburg, Gräfe und Sillem, 1893. 8vo. 8 and 176 pp. Illustrated. Boards. Mk. 2.20

Zwicky (M.). Grundriss der Planimetrie und Stereometrie nebst Uebungsaufgaben. Theil I: Planimetrie. Bern, Schmid, 1893. 8vo. 5 and 88 pp. Illustrated. Boards. Mk. 1.50

III. APPLIED MATHEMATICS.

Clapeyron (E.). Ueber die bewegende Kraft der Wärme. Deutsch herausgegeben von R. Mewes. Berlin, A. Friedländer, 1893. 8vo. 48 pp. Mk. 1.60

Crandall (C. L.). The transition curve, by offsets and deflection angles. London, 1893. 12mo. Mor. 7s. 6d.

Daltrop (V.). Die Gesetze des Fluges. Breslau, Max, 1893. 8vo. 44 pp. Illustrated. Mk. 0.80

Dorr (R.). Praktisch ausführbare Lösung des Problems der beliebigen Winkeltheilung. A. Beschreibung des Winkeltheilungsverfahrens, das ohne Anwendung eines besonderen Instruments auszuführen ist. B. Beschreibung zweier für das deutsche Reich patentirter Winkeltheilungsinstrumente. Elbing, Meissner, 1893. 8vo. 14 pp. Illustrated. Mk. 3.00

Engesser (F.). Die Zusatzkräfte und Nebenspannungen eiserner Fachwerkbrücken. Eine systematische Darstellung der verschiedenen Arten, ihrer Grösse und ihres Einflusses auf die konstruktive Gestaltung der Brücken. Theil II: Die Nebenspannungen. Berlin, Springer, 1893. 8vo. 6 and 191 pp. Illustrated. Mk. 7.00

Gauss (F. G.). Polygonometrische Tafeln, zum Gebrauche in der Landmessung für die Teilung des Quadranten in 90 Grade zu 60 Minuten. Halle, Strien, 1893. 8vo. 194 and 28 pp. Cloth. Mk. 12.00

Gottschalk (A.). Conjugierte Poinsot-Bewegungen. [Diss.] Münster, 1893. 8vo. 29 pp.

Huyghens (C.). Abhandlung über die Ursache der Schwere. Deutsch von R. Mewes. Berlin, A. Friedländer, 1893. 8vo. 10 and 47 pp. Illustrated. Mk. 1.60

Janusche (H.). Der Aetherdruck als einheitliche Naturkraft. [Progr.] Teschen, Prochaska, 1893. 8vo. 68 pp. Illustrated. Mk. 1.80

Keck (W.). Vorträge über Elasticitätslehre als Grundlage für die Festigkeits-Berechnung der Bauwerke. (2 parts.) Hannover, Helwing, 1893. 8vo. 14 and 361 pp. Illustrated. Mk. 10.00

Loney (S. L.). Solutions of the examples in the "Elements of Statics and Dynamics." London and New York, Macmillan, 1893. 16mo. 332 pp. Cloth. $1.75

McAulay (A.). Utility of quaternions in physics. London and New York, Macmillan, 1893. 8vo. 14 and 108 pp. Cloth. $1.60

Maiss (E.). Zur Lehre von den krummlinigen Bewegungen eines materiellen Punktes. [Progr.] Wien, 1893. 8vo. 24 pp. Illustrated.

Neumann (C.). Die Haupt- und Brenn-Punkte eines Linsen-Systemes. Elementare Darstellung der durch Möbius, Gauss und Bessel begründeten Theorie. 2d ed. Leipzig, Teubner, 1893. 8vo. 8 and 42 pp. Illustrated. Mk. 1.20

Ott (K. von). Vorträge über Baumechanik, gehalten an der k. k. deutschen technischen Hochschule in Prag. Theil III. Prag, Dominicus, 1893. 8vo. 14 and 400 pp. Illustrated. Mk. 10.40

Paoloni (P.). Disegno assonometrico. Milano, Hoepli, 1894. 16mo. 122 pp. Illustrated.

Poincaré (H.). Vorlesungen über Thermodynamik, redigirt von J. Blondin. Deutsche Ausgabe von W. Jaeger und E. Gumlich. Berlin, Springer, 1893. 8vo. 16 and 298 pp. Illustrated. Mk. 10.00

Rebber (W.). Die Festigkeitslehre und ihre Anwendung auf den Maschinenbau, elementar behandelt. 2d ed. Herausgegeben von L. Hummel. Mitweida, Schulze, 1893. 8vo. 10 and 402 pp. 22 plates. Mk. 10.50

Taylor (D. W.). Resistance of ships and screw-propulsion. With diagrams and tables. New York, Macmillan, 1893. 8vo. 234 pp. Cloth. $3.70

Tisserand (F.). Traité de mécanique céleste. Vol. III: Perturbations des planètes d'après la méthode de Hansen; théorie de la lune. Paris, Gauthier-Villars, 1894. 4to. Fr. 22.00

Watson (H. W.). A treatise on the kinetic theory of gases. 2d ed. London, Frowde, 1893. 8vo. 96 pp. 4s. 6d.

Weber (W.). Werke, herausgegeben von der königl. Gesellschaft der Wissenschaften zu Göttingen. Vol. V: Wellenlehre; besorgt durch E. Riecke. Berlin, Springer, 1893. 8vo. 30 and 438 pp. 18 plates. Mk. 18.00

Wenzel (L.). Ein Beitrag zur Schwingungstheorie elastischer Saiten. [Progr.] Klagenfurt, 1893. 8vo. 30 pp. Illustrated.

Wittmann (W.). Statik der Hochbauconstructionen. Theil II: Holz- und Eisenconstructionen. 2d ed. München, Rieger, 1893. 8vo. 8 and 328 pp. 3 plates. Illustrated. Mk. 12.00

Ziwet (A.). An elementary treatise on theoretical mechanics. Part I: Kinematics. New York, Macmillan, 1893. 8vo. 8 and 181 pp. Cloth. $2.25

A DOUBLY-INFINITE SYSTEM OF SIMPLE GROUPS.

(*Abstract of a Paper* presented to the Congress of Mathematics at Chicago, August* 25, 1893.)

BY PROF. E. HASTINGS MOORE.

1. *List of orders of systems of simple groups.*

The following is, so far as I know, a complete list of the orders of systems of simple groups which have as yet been determined (q = prime, n = positive integer).

(1) q.

(2) $\frac{1}{2}n!, \qquad (n > 4).$

(3) $\frac{1}{2}q(q^2 - 1), \qquad (q > 3);$

the group of the modular equation for the transformation of elliptic functions of order q.

(3′) $\frac{1}{2}q^n(q^{2n} - 1), \qquad (q > 2, (q, n) \neq (3, 1)),$

$q^n(q^{2n} - 1), \qquad (q = 2, n > 1).$

This system of simple groups is a generalization of the system (3) to be explained in this paper.

(4) $$\frac{(q^n - 1)q^{n-1}(q^{n-1} - 1)q^{n-2} \ldots (q^2 - 1)q}{\delta},$$

$((q, n) \neq (2, 2), (3, 2)),$

in which $\delta = [n, q - 1]$, the greatest common divisor of n and $q - 1$.

(5) $\frac{1}{2}(q^{2n} - 1)q^{2n-1}(q^{2n-2} - 1)q^{2n-3} \ldots (q^2 - 1)q, \qquad (q > 2),$

$(q^{2n} - 1)q^{2n-1}(q^{2n-2} - 1)q^{2n-3} \ldots (q^2 - 1)q,$
$(q = 2, n > 2).$

(6) $(P_n - 1)2^{2n-2}(P_{n-1} - 1)2^{2n-4} \ldots (P_2 - 1)2^2, \qquad (n > 2),$

in which $P_n = 2^{2n-1} + 2^{n-1}$.

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* This paper will be published in full in the Proceedings of the Congress. It will also appear as the first part of a paper, "The sub-groups of the simple group whose order is $\frac{1}{2}q^n(q^{2n} - 1)$ if $q > 2$, or $q^n(q^{2n} - 1)$ if $q = 2$," to be published in the *Mathematische Annalen*.

The systems* (4), (5), (6) are either given by Jordan or are derived from Jordan's decompositions of certain linear groups by the principle that the quotient-group† of any two consecutive groups in the series of composition of any group is a simple group.‡

The systems (1), (2), (3), (6) are simply infinite; the systems (3′), (4), (5) are doubly infinite. It is clear that of the three doubly-infinite systems the new system (3′) is the densest, that is, its orders increase least rapidly as q and n increase.

Professor Cole discovered last spring a new simple group of order 504 not contained in the six systems (1), (2), (3), (4), (5), (6). The facts (a) that in the system (3′) the group having $(q, n) = (3, 2)$, order 360, had previously been identified as holoedrically isomorphic with the alternating group in six letters (a simple group), and (b) that the group having $(q, n) = (2, 3)$ had the order 504 of Cole's new simple group, led to the present investigation.

The simple groups of composite order < 660 have been completely enumerated by Hölder (*Math. Annalen*, vol. 40) and Cole (*Amer. Journal of Math.*, vol. 14; BULLETIN OF NEW YORK MATH. SOCIETY, vol. 2, p. 254, foot-note). They are one group each for the orders 60, 168, 360 and 504. These are all included in the new system (3′), being the groups having, respectively, $(q, n) = (5, 1)$ or $(2, 2)$, $(7, 1)$, $(3, 2)$, and $(2, 3)$.

2. *The simply-infinite system* (3) *of simple groups of order* $\frac{1}{2}q(q^2-1)$, $(q > 3)$.

The formula

$$\omega' \equiv \frac{\alpha\omega+\beta}{\gamma\omega+\delta} \pmod{q},$$

where

$$\alpha\delta - \beta\gamma \equiv 1 \pmod{q}$$

and where $\alpha, \beta, \gamma, \delta$ are integers taken modulo q, and ω, ω' run through the series of $q+1$ values $0, 1, 2 \ldots \overline{q-1}, \infty$, may be considered an analytic expression of a certain substitution on the $q+1$ symbols or marks $0, 1, 2 \ldots \overline{q-1}, \infty$. The totality of all such distinct substitutions constitutes a group of order $\frac{1}{2}q(q^2 - 1)$, a particular form of the (abstract) group in question. The group is for every $q > 3$ simple. For an admirable exposition of the properties of this group, together with further references, see Klein-Fricke, "Modulfunctionen," vol. I, pp. 419–491.

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* See Jordan, "Traité des Substitutions," (4) p. 106, (5) pp. 176, 178, (6) pp. 205, 213. These references were given by Professor Cole in the paper, "On simple groups," presented by him to the Congress.

† See Hölder, *Math. Annalen*, vol. 34.

‡ Two systems of order (6) are given.

The existence and properties of the abstract group as studied under this form depend above all things upon

(a) The existence of the system of q marks, $0, 1, 2, \ldots q - 1$, which may be combined by the four fundamental operations of algebra, and in which the $q - 1$ marks (0 excluded) are given as the successive powers of one of them (a primitive congruence-root, modulo q);

(b) The introduction of the mark ∞ (due to Galois).

3. *The Galois-field of order* $s = q^n$.

Suppose we have a system of symbols or *marks*, $\mu_1, \mu_2 \ldots \mu_s$, in number s, and suppose that these s marks may be combined by the four fundamental operations of algebra—addition, subtraction, multiplication, and division—the operations being subject to the ordinary abstract operational laws of algebra ($\mu_f + \mu_g = \mu_g + \mu_f$, $\mu_f\mu_g = \mu_g\mu_f$, etc.), and that when the marks are so combined the results of these operations are in every case uniquely determined and belong to the system of marks. Such a system of s marks we call *a field of order s*.

The most familiar instance of such a field, of order $s = q =$ a prime, is the system of q incongruous classes (modulo q) of rational integral numbers a.

Galois discovered an important generalization of the preceding field. Let $F_n(\xi) = \sum_{0}^{n}{}^{i} c_i\xi^i$, where the c_i are integers and $c_n = 1$, be irreducible, modulo q. Then the Galois-field of order $s = q^n$, $GF[q^n]$, consists of the system of q^n incongruous classses (modulis q, $F_n(\xi)$) of rational integral functions of ξ with integral coefficients. In this $GF[q^n]$ there exist *primitive roots;* the $q^n - 1$ successive powers of a primitive root are the $q^n - 1$ marks of the field (0 excluded). The $GF[q^n]$ is *uniquely defined* for every $q =$ prime, $n =$ positive integer; that is,

(1) $F_n(\xi)$ which are irreducible (modulo q) do exist;

(2) The $GF[q^n]$ is independent of the particular $F_n(\xi)$ used in its construction.

For the details of this Galois theory, see Serret, "Algèbre supérieure," 5th edition, vol. II. pp. 122–189, and Jordan, "Substitutions," pp. 14–18.

It should be remarked further that every field of order s is in fact abstractly considered a Galois-field of order $s = q^n$.

4. *The doubly-infinite system* (3′) *of simple* groups of order* $\frac{1}{2}q^n(q^{2n}-1)$ if $q > 2$, *or* $q^n(q^{2n}-1)$ *if* $q = 2$.

Let α, β, γ, δ be marks of the $GF[q^n]$ satisfying the equation $\alpha\delta - \beta\gamma = 1$. Let ω, ω' be variable marks, each assuming, besides the q^n values of the $GF[q^n]$, the value ∞. The formula

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

expresses a certain substitution on the $q^n + 1$ marks ω. The totality of all such distinct substitutions constitutes a group of order $\frac{1}{2}q^n(q^{2n} - 1)$ if $q > 2$ or $q^n(q^{2n} - 1)$ if $q = 2$, a particular form of the (abstract) group in question.

For brevity we write hereafter s for q^n and $M(s) = M(q^n)$ for the order. I prove in this paper that this group $G_{M(s)}$ is a simple group in all except the two particular cases

$$\left\{\begin{array}{l} n = 1,\ q = 2,\ s = 2,\ M(s) = 6 \\ n = 1,\ q = 3,\ s = 3,\ M(s) = 12 \end{array}\right.$$

when the $G_{M(s)}$ are in fact known to be

the $G_{6 = 3!}$ symmetric substitution-group on $s + 1 = 3$ letters,
the G_{12} tetraedron group or alternating substitution-group on $s + 1 = 4$ letters,

and to have as self-conjugate sub-groups

a G_3 cyclic-group.
a G_4 four-group.

To this end it is necessary *first* to discuss the individual operators and the cyclic and commutative† sub-groups of the $G_{M(s)}$, and *secondly* to establish a diophantine equation for the order of a self-conjugate sub-group, which shall lead to the conclusion that the only self-conjugate sub-groups of the $G_{M(s)}$ are the identity and the $G_{M(s)}$ itself.

5. *The individual operators and the cyclic and commutative sub-groups of the* $G_{M(s)}$.

By an investigation differing somewhat in details from that of Serret‡ and Gierster§ for the case $n = 1$, as given in Klein-Fricke, "Modulfunctionen," vol. I, pp. 419–450, it is found that *every substitution* (the identity excepted) *determines and lies in one and only one largest commutative sub-group.* These sub-groups may be arranged in three different sets, the groups of each set being conjugate with one another under the $G_{M(s)}$.

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* $(q, n) = (2, 1)$, $(3, 1)$ excepted.

† A group is called *commutative* if its operators are commutative.

‡ Serret: *Comptes Rendus*, 1859, 1860; "Algèbre supérieure," vol. II, p. 363 ff.

§ Gierster: *Math. Annalen*, vol. 18, pp. 319-365; "Die Untergruppen der Galois'schen Gruppe der Modulargleichungen für den Fall eines primzahligen Transformationsgrades."

I. $s+1$ conjugate commutative $G_{s=q^n}$. These s^2-1 substitutions are all of period q, and for $q=2$ they are all conjugate, while for $q>2$ they separate into two sets of $\frac{1}{2}(s^2-1)$ conjugate substitutions. The $q-1$ substitutions of a cyclic group G_q belong

$$\text{for } q>2 \left\{ \begin{array}{l} n \text{ odd, half to one and half to the other} \\ n \text{ even, all to the same} \end{array} \right\} \text{ set of}$$

conjugate substitutions.

II. $\frac{1}{2}s(s+1)$ conjugate cyclic groups $G_{\frac{s-1}{2}}$ if $q>2$, or G_{s-1} if $q=2$.

III. $\frac{1}{2}s(s-1)$ conjugate cyclic groups $G_{\frac{s+1}{2}}$ if $q>2$, or G_{s+1} if $q=2$.

6. *The diophantine equation for the order of a self-conjugate sub-group of the $G_{M(s)}$.*

Let G_d be a self-conjugate sub-group of the $G_{M(s)}$. If it contains one of a set of conjugate substitutions or sub-groups of the $G_{M(s)}$, it will contain all of that set. Whence the G_d contains $\frac{1}{2}s(s+1)$ conjugate G_{d_-} from the conjugate groups (II), and $\frac{1}{2}s(s-1)$ conjugate G_{d_+} from the conjugate groups (III).

Further, the G_d contains either no substitution of period q, or

$$\left\{ \begin{array}{l} q>2,\ n \text{ odd, all of both sets} \\ q>2,\ n \text{ even, all of one set or of both sets} \\ q=2, \text{ all of the one set} \end{array} \right\} \text{ of conjugate}$$

substitutions of period q.

The enumeration of the substitutions of the self-conjugate G_d leads to the diophantine equation

$$-\tfrac{1}{2}(s^2-1)h+\tfrac{1}{2}s(s+1)d_-+\tfrac{1}{2}s(s-1)d_+=d,$$

to be satisfied for positive integral values of d, d_-, d_+, h, where d, d_-, d_+ are divisors of $\frac{1}{2}s(s^2-1)$, $\frac{1}{2}(s-1)$, $\frac{1}{2}(s+1)$, respectively, if $q>2$, or of $s(s^2-1)$, $s-1$, $s+1$, respectively, if $q=2$, and where h has the value

$$\left\{ \begin{array}{l} h=2 \text{ or } 0, \text{ for } q>2,\ n \text{ odd, and for } q=2, \\ h=2,\ 1 \text{ or } 0, \text{ for } q>2,\ n \text{ even.} \end{array} \right.$$

From this equation follows the simplicity of the $G_{M(s)}$ in all except the two particular cases $(q, n) = (2, 1)$ and $(3, 1)$.

THE UNIVERSITY OF CHICAGO, *October* 18, 1898.

NOTE ON MONOGENIC FUNCTIONS OF A SINGLE VARIABLE.

BY PROF. T. CRAIG.

THE following remark is so obvious that it seems impossible that it has not been made before; still neither the writer nor those to whom he has spoken have seen it.

Suppose $P(x, y)$, $Q(x, y)$ to be real functions of the real variables x, y. Form $P + iQ$: in order that this shall be a monogenic function of $x + iy = z$ it is necessary first that P and Q be functions satisfying Laplace's equation

$$\nabla P = \frac{\partial^2 P}{\partial x} + \frac{\partial^2 P}{\partial y^2} = 0,$$

$$\nabla Q = \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = 0.$$

But it is not sufficient that P and Q satisfy this equation. If P be a solution of the equation, Q must be determined by aid of Cauchy's equations

$$(1) \qquad \begin{aligned} \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} &= 0, \\ \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} &= 0; \end{aligned}$$

that is, Q will be given by the integral

$$Q = \int_{(x_0, y_0)}^{(x, y)} - \frac{\partial P}{\partial y} dx + \frac{\partial P}{\partial x} dy,$$

in which the condition of integrability is satisfied, since $\nabla P = 0$.

I say, however, that a monogenic function of $x + iy$ can be formed by aid of *any* two solutions of Laplace's equation and without any quadrature. Suppose P and Q to be any two such solutions; they do not then in general satisfy equations (1). If they do satisfy (1), then $P + iQ$ is the function sought. If they do not satisfy (1), write

$$Q_1 = \frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y},$$

$$P_1 = \frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}.$$

Then $P_1 + iQ_1$ is a monogenic function of $x + iy$, for

$$\frac{\partial P_1}{\partial x} - \frac{\partial Q_1}{\partial y} = 0,$$

$$\frac{\partial P_1}{\partial y} + \frac{\partial Q_1}{\partial x} = 0,$$

$$\nabla P_1 = \nabla Q_1 = 0,$$

$$\text{since } \nabla P = \nabla Q = 0.$$

LAMBERT'S NON-EUCLIDEAN GEOMETRY.

BY PROF. GEORGE BRUCE HALSTED.

In the discussion which followed my lecture in the Mathematical Section of the Congress at Chicago, Professor Study of Marburg mentioned that there had recently been brought to light an old paper of Lambert's on what was long after named the non-Euclidean geometry. Professor Klein jotted down on my Lobatschewsky programme the address of Dr. Staeckel, as the person from whom I might hope for definite information; and from his answer to my letter I extract the following highly interesting facts.

This essay of Lambert's bears the title: "Zur Theorie der Parallellinien." It is dated September, 1766, but was first published in 1786 from the papers left by F. Bernoulli, a relative of John Bernoulli. It appeared in the

"*Leipziger Magazin für reine und angewandte Mathematik;* herausgegeben von J. Bernoulli und C. F. Hindenburg, erster Jahrgang 1786. S. 13 ff." On account of the extraordinary interest of this article and the present great rarity of this magazine, the Leipziger Gesellschaft der Wissenschaften, at the suggestion of Dr. Staeckel, will reprint it in their *Abhandlungen*. Meanwhile, to show how wonderfully it anticipates Lobatschewsky, Bolyai, Riemann, Beltrami, in what it maintains, we need only cite the following:

(1) The parallel-axiom needs proof, since it does not hold for geometry on the surface of the sphere.

(2) In order to make intuitive a geometry in which the triangle's angle-sum is less than two right angles we need an "imaginary" sphere [pseudo-sphere].

(3) In a space in which the triangle's angle-sum is different from two right angles, there is an absolute measure [a natural unit for length].

The whole paper is another unexpected illustration of the words in a letter from Sir Robert Ball: "It is also noteworthy how many mathematicians, approaching the subject from very varied sides, have been led to the study of what mathematics would be like without the eleventh axiom."

AUSTIN, TEXAS, *November*, 1893.

THE TEACHING OF MATHEMATICS AT GÖTTINGEN.*

THE purpose of the following remarks is not to furnish to students who wish to prepare themselves to teach mathematics and physics in the higher schools, a detailed scheme of the lectures and exercises which they should attend during each semester. It would be impossible to do this on account of the great number of branches of mathematics and mathematical physics and the frequent changes necessary in the subjects and the arrangement of the courses of lectures. It is essential, however, that students should be acquainted with certain general principles and requirements of their chosen subject in order to bring their course of study into conformity with them, and that they should have a general idea of the whole field to be covered in order to distribute their lectures and work conveniently through the several semesters. It is to be observed that in the theoretical development of mathematics and mathematical physics the advanced courses rest altogether upon foundations laid in the elements, so that after the introductory courses of the first two or three semesters have been attended and are properly understood, the advanced courses, so far as they are not immediate continuations of previous ones, may be taken in almost any order.

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* Translation of a circular, "Announcement of a scheme of study for those wishing to become teachers of mathematics and physics; together with an extract from the regulations of the seminary of mathematics and physics," issued by the University, and reprinted in the *Zeitschrift für math und naturwiss. Unterricht*, vol. 24. pp. 540-546. Apart from its relation to the work at Göttingen, the circular is of general interest to teachers of mathematical science.—T. S. F.

The following advice to our students relates in its "general part" to the

1. Introductory lectures of the first semester.
2. Elaboration of the lectures.
3. Number of courses to be attended.
4. Subordinate subjects.
5. Philosophical courses.
6. Work in the seminary and laboratories.
7. Private study.

In a second "special part" is given an account of the most important courses of lectures, with remarks upon several groups of them.

I. General Part.

1. *Introductory lectures of the first semester.*

Analytical geometry and the differential and integral calculus, including algebraic analysis, constitute the foundations not only of the higher mathematics in its narrower sense but also of mechanics and mathematical physics. They should therefore be completed, as far as possible, in the first two semesters of study. The same is true of the course in experimental physics, knowledge of which is required for attendance upon the practical work in physics and the study of mathematical physics. As a preparation for the latter, mechanics is also necessary, and we would recommend its study in the third semester. For further details we refer to the "special part" of this plan of study.

2. *Elaboration of the lectures.*

It cannot be stated with sufficient emphasis that mere listening to a course of lectures has no value in mathematics or mathematical physics, and but little value in experimental physics. The student can make the material presented to him in the lecture his own property only by carefully elaborating at home the notes which he has taken down during its delivery. In this connection he is earnestly requested to consult the lecturer whenever in difficulty or in doubt. Such questions, which will materially lighten the labor of elaboration, are much desired, and will always be most willingly answered.

3. *Number of courses to be taken.*

The requirement just mentioned limits the number of courses in mathematics and mathematical physics which can be attended simultaneously. It is not possible to elaborate together, nor to attend therefore with profit, more than two, or at most three, four-hour mathematical courses; particularly as courses in natural science and other subjects are also to be attended. The necessity of choosing from the whole number of courses offered, those which are indicated by the special direction of the student's personal interest, will make itself more or less felt in the later semesters. The thorough study of some special branch of science, for which the University of Göttingen through its great number of special courses affords peculiarly favorable opportunities, is much to be encouraged on account of the resulting thoroughness of scientific training. Such a concentration of study in a special department can be accomplished without interfering with the demands of the examination, which requires for teachers of the higher classes in mathematics "a well grounded knowledge of the most important theories of higher geometry, higher analysis and analytical mechanics," and in physics "a general survey of mathematical physics, and an acquaintance with the fundamental mathematical investigations in one of the more important departments of theoretical physics."

4. *Subordinate subjects.*

No general rule can be given in regard to the subordinate subjects (natural sciences, geography, etc.), on account of the diversity of the students' aims and inclinations. It will generally be in the interest of the mathematical and physical studies to attend these subjects in the earlier semesters, in order to keep the time in the later semesters as free as possible for the seminary work and advanced lectures.

5. *Philosophical lectures.*

The philosophical lectures may be postponed with advantage to the second half of the period of studentship. They should not present to the student a theoretical system of essentially dogmatic character, but rather should enable him to trace out, from the point of view of his own subject, the relations which philosophy bears to an extensive field of positive knowledge.

6. *Work in the seminary and laboratories.*

The lectures on mathematics and natural science are supplemented by courses devoted to exercises and practical work, from which the students gain, in addition to knowledge, a corresponding facility in its use. Most of the mathematical exercises are held in the seminarium of mathematics and physics. The student is earnestly recommended to enter the seminary during his earlier semesters, since even at this period it affords, through its reading-room, great assistance and encouragement to study. The conditions for admission to and the use of the reading-room are mentioned below.

The exercises in descriptive geometry should be taken, as a rule, in the first half of the period of studentship, which is less burdened with other practical work. Practice in geometrical drawing is so important a requirement for the teacher of mathematics and physics that no student should lose the opportunity offered for acquiring it.

The practical work in physics and chemistry attach themselves to the corresponding lectures, and should therefore be performed in general during the middle semesters. In physics and chemistry it has been long established by experience, that mere attendance upon lectures is insufficient for a thorough understanding, and that work of observation and measurement in the laboratories is indispensable. In mathematics, also the application of the general principles to particular examples and practice in the performance of numerical calculations form an important auxiliary to theoretical progress. In this connection, the attention of the students is called to the work in magnetism, astronomy, and geodesy, in which, quite apart from the general interest of the subjects treated, instruction is afforded in the systematic reduction of extended series of observations.

7. *Private study.*

The information acquired by lectures and practical work should be completed, primarily by private study of textbooks. Individual mental activity will be developed in a greater degree, however, by reading original works. Facility in appreciating new ideas will be thus obtained, and upon this depends the ability to follow, in later years, the subsequent developments of science. The general library of the university and the reading-room of the seminarium of mathematics and physics, both of which are open during vacation, furnish the best opportunity for an extensive study of the literature.

We finally urge upon our students that they should not lose sight of their future occupations. The lectures attended by them afford many opportunities for reflecting upon the subjects of study in the gymnasia, and for appreciating the significance which mathematics and physics possess for the whole of our modern civilization. The breadth of view thus to be gained will be of great assistance in quickening the instruction to be imparted later, and in rendering it suggestive and productive. For the same reason, those courses are of the greatest importance to future teachers which relate especially to the subject of elementary mathematics. These courses are intended for the later semesters.

II. Special Part.

A. *Courses in mathematics and mathematical physics.*

The number of courses in mathematics and mathematical physics at Göttingen is unusually great. Not only are those subjects treated which, in the present state of science, have a recognized place in academic instruction; but numerous courses extend into those special departments of science which have only recently been established and are still actually in process of construction. It thus appears the more necessary to append to the general remarks of the preceding part, an account of the courses of lectures themselves. These are divided into introductory courses, advanced courses, and special courses. The introductory courses are given every year; the advanced courses are repeated every two or three years; while in the case of the special courses there is no regular rotation.

In the following table the title of the course first given is the most general for each subject. The other titles, added in parentheses, relate to courses which possibly have special aims, but are on the whole equivalent to that first specified.

Introductory lectures.

(1) Analytical geometry.

(2) Differential and integral calculus.

(3) Mechanics. (Elementary mechanics, Introduction to analytical mechanics, Introduction to theoretical physics.)

The differential and integral calculus extends at times through two semesters. Occasionally a special course in algebraic analysis is also given, in which certain parts of algebra are treated by themselves.

Sometimes a course is given under the title "Introduction to the higher mathematics." This is intended for students of other subjects, who wish to acquire a limited knowledge of higher mathematics. Such a course does not afford a sufficient basis for a further study of mathematics.

Advanced lectures.

(1) Algebra. (Theory of algebraic and numerical equations, Determinants, Invariants.)

(2) Theory of numbers.

(3) Higher geometry. (Analytical and synthetic treatment of projective geometry.)

(4) Curved surfaces, and curves of double curvature. (Application of the differential and integral calculus to geometry.)

(5) General survey of the elementary mathematics.

This course, in which at one time the algebraic, at another time the geometrical side is put in the foreground, has for its object the elucidation of the principles of elementary mathematics from the point of view of the higher mathematics.

(6) Higher parts of the integral calculus. (Definite integrals, Differential equations, Calculus of variations.)

(7) Theory of functions.

(8) Elliptic functions.

The three courses immediately preceding are of fundamental importance, not only for all branches of pure mathematics, but for their applications to mathematical physics.

(9) Higher mechanics. (Analytical mechanics.)

Higher mechanics treats largely of analytical problems developed by the necessities of astronomy. The importance of the geometrical element is, however, constantly increasing on account of the technical applications. (Graphical statics, Kinematics.)

(10) Potential.

Potential will be given as a rule during two semesters, and introduces the applications of the theory of potential to gravitation, electrostatics, magnetism, electrodynamics, and hydrodynamics.

(11) Partial differential equations.

This course is devoted to the methods of integrating the differential equations of mathematical physics.

(10′) General theory of electricity.

(11′) Theory of elasticity, and acoustics.

Courses 10, 11 on the one hand, 10′, 11′ on the other, treat in great part of the same subjects, the former more from the mathematical, the latter more from the physical, point of view. They are therefore to a certain extent equivalent.

(12) Theory of light.

(13) Theory of heat. (Mechanical theory of heat, Chemical physics.)

In the courses in mathematical physics not only the theory of the physical phenomena presented in the course in experimental physics is elaborated, and the methods of their measurement given, but many important results, which to be understood require previous theoretical training, are stated for the first time.

(14) General astronomy. (Geodesy, Astrophysics, Geophysics.)

(15) Theory of probabilities. (Method of least squares.)

Apart from its general value in reasoning, the theory of probabilities possesses great importance, on the one hand for the method of least squares, indispensable to the sciences depending upon measurement, and on the other, for the scientific theory of statistics and life insurance.

Special courses.

The special courses are intended chiefly for those students who have inclination or occasion for independent scientific work, such as, among other things, is necessarily assumed in the case of candidates for the doctor's degree.

The subjects of these courses cannot be specified as they change with the state of science and with the predilections of the lecturers. In general, special parts of the preceding advanced lectures will be treated more in detail.

B. *The philosophical courses.*

1. History of Philosophy. 2. Logic. 3. Psychology. 4. Pedagogics.

In this place it is necessary to remind the students that they should not neglect their general culture during the period of their studentship. They should consider, therefore, in addition to the philosophical courses, those in history and in the history of literature.

C. *Courses in natural science and geography.*

1. Experimental physics.

The course in experimental physics extends through two semesters, and is so arranged that the second part of the course can be taken without previous attendance upon the first part.

2. Experimental chemistry.

Here the first part of the course, inorganic chemistry, is indispensable to the understanding of the second part, organic chemistry. Attendance upon the practical work in chemistry is permissible during the course in inorganic chemistry.

3. Mineralogy. (Crystallography.)

Apart from its individual importance, this course contains important material complementary to certain parts of physics and chemistry.

4. Zoology. 5. Botany.

In regard to botany it may be stated that those students who wish to teach it in the middle classes must not fail to make themselves acquainted with the local flora by participating in the botanical excursions.

6. Geography. 7. Geology.

Physical geography bears close relations to all branches of natural science and depends, in a variety of ways, upon mathematics and physics. Geology is complementary to physical geography, and knowledge of it is necessary for understanding more fully the laws of morphology and of geographical distribution. Geological excursions are a necessary auxiliary to the course in geology.

Extract from the regulations of the seminary of mathematics and physics.

1. The province of the seminary is to lead the students to independent work and to instruct them in the application of what they have learned in the lectures.

2. The object of the seminary is accomplished by discourses on the part of the directors and by exercises performed under their guidance. The latter consist of lectures delivered by the students themselves, and the solution of problems assigned by the director to be discussed in writing. The students are given opportunities also to make physical, magnetic and astronomical measurements, in the performance of which the rooms and instruments of the physical institute and the astronomical observatory may be employed. The lectures and exercises of the seminary of mathematics and physics are regularly announced in the scheme of lectures.

3. Every matriculated student of mathematics and physics may become a member of the seminary. Those entering must make themselves known to the director in charge, and, after admission, must present themselves to all the other directors whose work they wish to attend.

4. The reading-room of the seminary of mathematics and physics is designed to make the literature of these subjects as accessible as possible to the students in prosecuting their studies.

5. Those members of the seminary who wish to use the reading-room must make personal application to a director,

and must agree to observe the rules established for the management of the reading-room.

6. In accordance with a legal statute, a library fee, payable in advance, is collected from every member at the beginning of each semester. For ordinary members, to each of whom a private drawer in the reading-room may be assigned, the fee is five marks, and for other members it is three marks. In addition, there is a deposit of three marks required from each member receiving a key, and returnable upon the surrender of the key.

7. The collection of these fees and the maintenance of order in the reading-room are in charge of the librarian, under the supervision of the directors. Members are required to obey his instructions. Further regulations will be announced by means of notices in the reading-room.

The directors of the seminary of mathematics and physics: RIECKE, SCHERING, VOIGT, KLEIN, SCHUR, WEBER.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, November 4, at half-past three o'clock, the president, Dr. McClintock, in the chair. Mr. Isaac Hodges Turrell of Cincinnati, Ohio, having been duly nominated, and being recommended by the council, was elected a member. Professor Fletcher Durell read a paper entitled "Application of the new education to the differential and integral calculus," in which he advocated the presentation of the first principles of the calculus from an almost purely geometrical point of view. T. S. F.

THE National Academy of Sciences met in the Capitol at Albany, November 7–9. The papers presented included one by Dr. S. C. Chandler, entitled "Additional researches on the motion of the earth's pole." Dr. Chandler finds that the most recent observations obtainable (some still unpublished) confirm the law deduced by him. He showed that the two separate motions of the pole both take place from west to east. Dr. Chandler's paper was discussed by Professors Hall, Newcomb and Boss. They all expressed themselves as now favoring the truth of Dr. Chandler's law of variations. Professor C. S. Hastings read a paper on "A new form of telescopic objective, as applied to the twelve-inch equatorial of the Dudley Observatory." The principal characteristics of this instrument are: first, that one of the "ghosts" is made to coincide with the focal plane, thus rendering it harmless; and second, that the transformation from a visual to a photographic telescope is accomplished by substituting a second glass for one of the lenses of the visual combination, instead of adding a third lense. Professor Asaph Hall read a short paper on "Double stars." Professor Charles L. Doolittle (introduced by Professor Boss) communicated a paper, "Latitude determinations at the Sayre Observatory," but was unable to be present in person.

During the afternoon of November 8 the members of the Academy visited the new Dudley Observatory by invitation. The completed observatory was opened for inspection, and an address was made by Professor Simon Newcomb. H. J.

OBERLIN COLLEGE has secured for its library the collection of mathematical books sent by B. G. Teubner, of Leipzig, to the mathematical section of the German university exhibit at the Chicago Exposition. This collection contains over 150 important modern treatises, in addition to complete sets of the *Mathematische Annalen, Zeitschrift für Mathematik und Physik,* and *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht.*

THE annual meeting of the London Mathematical Society was held on November 9. The officers and council elected for the coming year are: President, A. B. Kempe; Vice-Presidents, A. B. Basset, Prof. Elliott, Prof. Greenhill; Treasurer, J. Larmor; Secretaries, M. Jenkins, R. Tucker; other members of the Council, Lieut.-Col. J. R. Campbell, Lieut.-Col. A. J. Cunningham, A. R. Forsyth, J. W. L. Glaisher, Prof. M. J. M. Hill, E. W. Hobson, A. E. H. Love, Major MacMahon, J. J. Walker. The De Morgan medal, which had been awarded to Professor Klein, was presented, in his absence, to Professor Greenhill and Dr. Forsyth, who had been appointed to receive it for him, and who made replies in his behalf.

A REPORT from the committee appointed to receive subscriptions for the Gauss-Weber memorial at Göttingen announces that the total amount of contributions will probably reach 25,000 marks.

DR. J. H. BOYD has accepted a position as tutor in the department of mathematics at the University of Chicago.
T. S. F.

PROFESSOR G. E. FISHER and Dr. Schwatt, of the University of Pennsylvania, are preparing an English translation of H. Durège's "Elemente der Theorie der Funktionen einer

complexen veränderlichen Grösse." The translation will be supplemented by a considerable amount of additional matter and by the insertion of numerous exercises. B. G. Teubner, of Leipzig, is the publisher. The book is expected next summer.
A. Z.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Binder (W.). Über Plancurven vierter Ordnung vom Geschlechte $p = 1$ und ihre typischen Formen. [Progr.] Wiener-Neustadt, 1893. 8vo. 64 pp. 1 plate.

Breijer (H. G.). De grondvormen der krommen van de 3e klasse. [Diss.] Amsterdam, 1893. 8vo. 8 and 72 pp.

Broecker (H.). Die Periodicitätsmoduln der Abel'schen Integrale erster Gattung als Functionen eines Parameters aufgefasst für den Fall einer Riemann'schen Fläche, deren Gleichung in Bezug auf die eine Variabele binomisch ist. Berlin, 1893. 8vo. 55 pp. 1 plate. Mk. 1.50

Dedekind (R.). Was sind und was sollen die Zahlen? 2te Auflage. Braunschweig, Vieweg, 1893. 8vo. 19 and 58 pp. Mk. 1.60

Fleury (H.). L'analyse infinitésimale sans limites ni infiniment petits. Paris, 1893. 8vo. 40 pp. Mk. 2.00

Goursat (E.). Vorlesungen über die Integration der partiellen Differentialgleichungen erster Ordnung, gehalten an der Faculté des Sciences zu Paris, bearbeitet von C. Bourlet. Autorisirte deutsche Ausgabe von H. Maser. Mit einem Begleitwort von S. Lie. Leipzig, Teubner, 1893. 8vo. 12 and 416 pp. Mk. 10.00

Harst (A. D. van der). Bijdrage tot de theorie en toepassing der kromlijnige coördinaten. [Diss.] Leiden, 1893. 4to. 10 and 55 pp.

Jahrbuch über die Fortschritte der Mathematik, begründet von C. Ohrtmann, herausgegeben von E. Lampe. Vol. 22 (Jahrgang 1890), 3tes Heft. Berlin, Reimer, 1893. 8vo. 68 pp. and pp. 849–1313. Mk. 13.00

Jochnick (W.). Det vigtigaste af analytiska Geometrien. Stockholm, 1893. 8vo. 136 pp. Mk. 5.00

Käsbohrer (L.). Die linear-unabhängigen quadratischen Relationen zwischen den zum Kurvengeschlecht $p = 3$ gehörigen Formen ϕ. [Progr.] Nürnberg, 1893. 8vo. 43 pp.

Killing (W.). Einführung in die Grundlagen der Geometrie. Band 1. Paderborn, 1893. 8vo. 10 and 357 pp. Illustrated. Mk. 7.00

Klein (F.). On Riemann's theory of algebraic functions and their integrals. A supplement to the usual treatises. Translated from the German with the author's permission by Frances Hardcastle. London and New York, Macmillan, 1893. 8vo. 76 pp. $1.40

Landsberg (G.). Zur Theorie der Gauss'schen Summen und der linearen Transformation der Thetafunctionen. (Habilitationsschrift.) Heidelberg, 1893. 4to. 22 pp.

Laurent (H.). Théorie des jeux de hasard. Paris, 1893. 8vo. Fr. 2 50

Lie (S.). Theorie der Transformationsgruppen. Dritter und letzter Abschnitt. Unter Mitwirkung von F. Engel bearbeitet. Leipzig, Teubner, 1893. 8vo. 27 and 831 pp. Mk. 26.00

——— Vorlesungen über continuirliche Gruppen mit geometrischen und anderen Anwendungen. Bearbeitet und herausgegeben von G. Scheffers. Leipzig, Teubner, 1893. 8vo. 15 and 810 pp. Illustrated. Mk. 24.00

Marz (L.). Ueber die Entstehung der Zahlwörter. [Progr.] Memmingen. 1893. 8vo. 6 pp.

Mukhopadhyay (A.). An elementary treatise on the geometry of conics. London, Macmillan, 1893. 8vo. 4s. 6d.

Repetitorium (Kurzes) der höheren Mathematik. Theil I: Differentialrechnung. Bearbeitet nach den Werken und Vorlesungen von Duhamel, Herr, Lipschitz, Salmon, Schlömilch und Anderen. Wien, Breitenstein, 1893. 8vo. 81 pp. Illustrated. Mk. 1.10

Rohn (K.) **und Papperitz** (E.). Lehrbuch der darstellenden Geometrie. (In 2 vols.) Vol. I. Leipzig, Veit, 1893. 8vo. 18 and 381 pp. Illustrated. Mk. 11.00

Scheffler (H.). Beleuchtung und Beweis eines Satzes aus Legendre's Zahlentheorie. Leipzig, Foerster, 1893. 8vo. 40 pp. Mk. 1.00

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MODERN MATHEMATICAL THOUGHT.

(ADDRESS DELIVERED BEFORE THE NEW YORK MATHEMATICAL SOCIETY AT THE ANNUAL MEETING, DECEMBER 28, 1893.)

BY PROF. SIMON NEWCOMB.

ONE who, like myself, is not a mathematician in the modern sense naturally feels that some apology is due for accepting the invitation with which this Society has honored me, to address it on a mathematical subject. Possibly an adequate apology may be found in the reflection that one who has not gone deeply into any of the contemporaneous problems of mathematics, but who, as a student, has had a sufficient fondness for the subject to keep himself informed of the general course of thought in it, may be able to take such a general review as is appropriate to the present occasion. I shall therefore ask your consideration of some comparisons between the mode of thinking on mathematical subjects at the present time, and those methods which have come down to us from the past, with a view of pointing out in what direction progress lies, and what is the significance of mathematical investigation at the present day.

Among the miscellaneous reading of my youth was a history of modern Europe, which concluded with a general survey and attempted forecast of progress in arts, science, and literature. So far as I can judge, this work was written about the time of Euler or Lagrange. On the subject of mathematics the writer's conclusion was that fruitful investigation seemed at an end, and that there was little prospect of brilliant discoveries in the future. To us, a century later, this judgment might seem to illustrate the danger of prophesying, and lead us to look upon the author as one who must have been too prone to hasty conclusions. I am not sure that careful analysis would not show the author's view to be less rash than it may now appear. May we not say that in the special direction and along the special lines which mathematical research was following a century ago no very brilliant discoveries have been made? Can we really say that Euler's field of work has been greatly widened since his time? Of the great problems which baffled the skill of the ancient geometers, including the quadrature of the circle, the duplication of the cube, and the trisection of the angle, we have not solved one. Our only advance in treating them has been to show that they are insoluble. To the problem of three bodies we have not added one of the integrals necessary to the complete solution. Our elementary integral calculus is two centuries old. For the general equation of the fifth degree we have only shown that no solution exists. We should, doubtless, solve many of the problems which the Bernoullis and their contemporaries amused themselves by putting to each other, rather better than they did; but, after all, could we get any solution which was beyond their powers? I speak with some diffidence on such a point as this; but it seems to me that progress has been made by going back to elementary principles, and starting out to survey the whole field of mathematical investigation from a higher plane than that on which our predecessors stood, rather than by continuing on the lines which they followed.

We may illustrate this passage to new modes of thought by comparing Euclid's doctrine of ratio and proportion with our own. No one questions the beauty or rigor of the process by which Euclid developed this doctrine in his fifth book, and applied it to the theory of numbers in his seventh book. But can we help pitying our forefathers who had to learn the complex propositions and ponderous demonstrations of the fifth book, all the processes and results of which we could now write on a single sheet of paper? As a mental discipline the study was excellent; but it seems hardly possible that one could have remembered the propositions or the methods of demonstrating them if he had no other knowledge of them than that derived from the work itself. When we carefully examine these propositions, we find that while Euclid recognized the fact that one of two ratios might be greater than, equal to, or less than another, yet he never regarded them as quantities which could be used as operands. From his standpoint a ratio was always a relation, and a relation cannot exist without two terms.

In pointing out this complexity of Euclid's doctrine, I must not be taken to indorse the very loose way in which the doctrine in question is usually treated in our modern text-books. What we should aim at is to replace Euclid's methods by those which pertain to modern mathematics. At the present time we conceive that a relation between any two concepts of the same kind may always be reduced to a single term by substituting for it an operator whose function it is to change one of these concepts into the other. In the case of the relation between two lines, considered simply as one dimensional quantities, which relation is called a ratio, we regard the ratio as a numerical factor or multiple, which, operating on one line, changes it into the other. For example, that relation which Euclid would have expressed by saying that two lines were to each other as 5 to 2, or that twice one line was equal to five times the other, we should now express by saying that if we multiplied one of the lines by two and one half, we should produce the other. This might seem to be simple difference of words, but it is much more. It is a simplification of ideas; a substitution of one conception for two. Euclid needed two terms to express a relation; we need but one.

But this is not the only simplification. A peculiarity of our modern mathematics is that operators themselves are regarded as independent objects of reasoning; susceptible of becoming operands, without specification of their particular qualities as operators. Thus, instead of considering the ratio which I have just mentioned as an operation of multiplying a line by two and one half, we finally reduce it to the simple quantity two and one half, which we may conceive to remain inert until we bring it into activity as a multiplier. It thus assumes a concrete form, capable of being carried about in thought, and operated upon as if it were a single thing.

This example may afford us a starting-point for a farther illustration of the way in which we have broadened the conceptions which lie at the basis of mathematical thought. Let us reflect upon the relation between a straight line going out from a certain point, and another line of equal length going out from the same point at right angles to the first. Had this relation been presented to Euclid as a subject of study, he would probably have replied that though much simpler than those he was studying, he could see nothing fruitful in it, and would have drawn no conclusions from it. But if we trace up the thought we shall find a wide field before us, embracing the first conception of groups, and with it an important part of our modern mathematics. In accordance with the principle already set forth, we replace the relation between these two lines by an operator which will change the first into the second. We define this operator by saying that its function is to turn a line through a right angle in a fixed plane containing the line. This definition permits of the operator in question being applied to any line in the plane. Then let us apply it twice in succession to the same line. The result will be a line pointing in the opposite direction from the original one. A third operation will bring it again to a right angle on the opposite side from the second position; and a fourth will restore the line to its original position, the result being to carry it through a complete circle. If we now consider the operations which would have been equivalent to these one, two, three, and four revolutions through a right angle as four separate operators, we see that their results will be either to leave the line in its original position, or to move it into one of three definite positions. If we then repeat any of these four operations as often as we please or in any order we please, we shall only bring the line to one of the four positions in question. We thus have a group of the fourth order, possessing the property that the repetition of any two operations of the group is equivalent to some single operation of it.

I scarcely need call attention to the familiar homology between these operations and successive multiplications by the imaginary unit. This last concept, considered as a multiplier, has the same properties as our rotating operator. Repeated twice, it changes the sign or direction of the quantity on which it operates; repeated four times, it restores it to its original value.

We have here a very simple illustration of a law of thought the application of which forms the basis of an important part of modern mathematical research. We may call it the law of homology. I am not sure of my ability to define it rigorously, but I think we may express it in some such form as this: If we have two sets of concepts, say A and B, such that to every concept of the one set shall correspond a concept of the other, and to every relation between any two of one set a corresponding relation between the corresponding two of the other, then all language, reasoning, and conclusions as to the one set may be applied to the other set. We may, of course, extend the law to a correspondence between things or concepts, and symbols or other forms of language.

This law is, I think, more universal than might at first sight appear. Not only the progress but the very existence of our race depends upon that co-ordination between our mental processes and the processes of the external universe, which has gradually been brought about by the attrition between man and nature through unnumbered generations. A man is perfect, powerful, and effective in proportion as his thoughts of nature coincide with the processes of nature herself; each process of nature having its image in his thought, and *vice versa*. Now language consists in a co-ordination between words and conceptions. Thus we pass from nature to what corresponds to it in thought, and from thought to what corresponds to it in language, and thus bring about a correspondence between language and nature.

Modern scientific research affords many examples of the application of this law, which would be very marvellous if they were not so familiar. We are so accustomed to the prediction of an eclipse that we see no philosophy in it. And yet might not a very intellectual being from another sphere see something wonderful in the fact that by a process of making symbols with pen and ink upon sheets of paper, and combining them according to certain simple rules, it is possible to predict with unerring certainty that the shadow of the moon, on a given day and at a given hour and minute, will pass over a certain place on the earth's surface? Surely the being might ask with surprise how such a result could be attained. Our reply would be simply this: There is a one-to-one correspondence between the symbols which the mathematician makes on his paper, and the laws of motion of the heavenly bodies. His symbols embody the methods of nature itself.

The introduction and application of homologies such as I have pointed out have, perhaps, their greatest value as thought-savers. In the field of mathematical thought they bear some resemblance to labor-saving machines in the field of economics. They enable the results of ratiocination to be reached without going through the process of reasoning in the particular case. Much that I have said illustrates this use of the method, but there is yet another case which has been so fruitful as to be worthy of special mention: I mean the general theory of functions of an imaginary variable. We may regard such functions as being in reality representative of a pair of functions of a certain class involving a pair of real variables; but the difficulty of conceiving the various ways in which the two variables might be related, and the results of the changes which they might go through, in such a way as to clearly follow out all possible results, would have rendered their direct study impossible.

But when Gauss and Cauchy conceived the happy idea of representing two such variables, the real and the imaginary one, by the rectangular co-ordinates of a point in a plane, those relations which before taxed the powers of conception became comparatively simple. Considered as a magnitude, the complex variable, or the sum of a real quantity and a purely imaginary one, the latter being considered as one measured in imaginary units, was represented by the length and position of a straight line drawn from an origin of co-ordinates to the point whose co-ordinates were represented by the values of the variable. Such a line, when both length and direction are considered, is now familiarly known as a vector. The conception of the vector would, however, in many cases be laborious. But the vector is completely determined by its terminal point; to every vector corresponds one and only one terminal point, and to every terminal point one and only one vector. Hence we may make abstraction of the vector entirely, and in thought attend only to the terminal point. Since for every pair of values we assign to our original variables there is one point, and only one, we may in thought make abstraction of both of these variables, and of the vectors which they represent, and consider only the point whose co-ordinates they are. Thus the continuous variation of the two quantities, how complex soever it may be, is represented by a motion of the point. Now such a motion is very easy to conceive. We may consider it as performing a number of revolutions around some fixed position without the slightest difficulty, whereas to conceive the corresponding variations in the algebraic variables themselves would need considerable mental effort. Thus, and thus alone, has the beautiful theory, first largely developed by Cauchy, and afterward continued by Riemann, been brought to its present state of perfection.

Another example of the principle in question, where the two objections of reasoning are so nearly of a kind that no thought is saved, is afforded by the principle of duality in projective geometry. Here a one-to-one correspondence is established between the mutual relations of points and lines, with the result that in demonstrating any proposition relating to these concepts we at the same time demonstrate a correlative proposition formed from the original one by simply interchanging the words "point" and "line."

The subjects of which I have heretofore spoken belong conjointly to algebra and geometry. Indeed, one of the great results of bringing homologous interpretation into modern mathematics has been to unify the treatment of algebra and geometry, and almost fuse them into a single science. To a large class of theorems of algebra belong corresponding theorems of geometry, each of one class proving one of the other class. Thus the two sciences become mutually helpful. In geometry we have a visible representation of algebraic theorems; by algebraic operations we reach geometrical conclusions which it might be much more difficult to reach by direct reasoning. A remarkable example is afforded by the geometrical application of the theory of invariants. These are perhaps the last kind of algebraic conclusions which the student, when they are first presented to his attention, would conceive to have a geometrical application. Yet a very little study suffices to establish a complete homology between them and the distribution of points upon a straight line.

This use of homologies does not mark the only line by which we have advanced beyond our predecessors. Progress has been possible only by emancipating ourselves from certain of the conceptions of ancient geometry which are still uppermost in all our elementary teaching. The illustration I have already given is here much to the point. The expression of a relation between two straight lines by the multiplier which would change one into the other is now familiar to every schoolboy, and the relation itself was familiar to Euclid. But the yet simpler relation of a line to another of equal length standing at right angles to it, and the corresponding operator which will change one into the other, was never thought of by Euclid, and is unfamiliar in our schools.

Why is this? It seems to me that it grows out of the ancestral idea that mathematics concerns itself with measurement, and that the object of measurement is to express all magnitudes in one-dimensional measure. So completely has this idea directed language, that we still extend the use of the word "equal" to all cases of this particular kind of linear equality: we say that a circle is equal to the rectangle contained by its radius and half its circumference. We have therefore been obliged to invent the word "congruent" for absolute equality in all points, or to qualify the adjective "equal" by "identical," saying "identically equal." There is of course no objection to the comparison of magnitudes in this way by reference to one dimensional measures, or by presupposing that the change which one magnitude must undergo in order to be transformed into the other is to be expressed by a single parameter, but changes involving two or any number of parameters, are just as important as those involving one, and the attempt to express all metric relations by referring them to a single parameter has placed such restrictions on thought that it seems to me appropriate to apply the term emancipation to our act in freeing ourselves from them. With us mathematics is no longer the science of quantity. But even if we consider that the ultimate object of mathematics is relations between quantities, we have reaped a rich reward by the emancipation, for we are enabled by the use of our broader ideas to reach new conclusions as to metric relations.

I trust that my discourse will not be found too academic if I continue with some further illustrations of the homologies of groups. Returning to the relations between lines, let us suppose that instead of taking two lines at right angles to each other, we consider two which form an angle of 10 degrees. As already remarked, this relation is homologous with an operator which will turn a single line through that angle. If we continually repeat this operation, we shall bring the line into 35 different positions, the 36th position being identical with the original one. Thus we should have 36 positions in all, expressed by that number of lines radiating from a single centre, and making angles of 10 degrees with each other. Now let us imagine 36 operators whose function it is to turn a line, no matter what, successively through an arc of 10 degrees, 20 degrees, 30 degrees, etc., up to 360 degrees, the last being equivalent to an operator which simply does nothing. These 36 operators will form a group which we know to be strictly homologous with multiplication by the 36 expressions

$$e^{i\phi},\ e^{2i\phi},\ e^{3i\phi}\ \ldots\ e^{36i\phi} = e^{0} = 1,$$

where ϕ is the arc of 10° in circular measure.

So far we have only considered operations formed by the continual repetition of a single one; in the language of the subject, all our groups are constructed from powers of a single operator. Now let us extend our process by substituting a cube for our straight line. Through this cube we have an axis parallel to four of its plane sides. By rotating the cube through any multiples of 90 degrees around this axis we effect an interchange of position between four of its sides. This process of interchanging is homologous with rotation through 90 degrees, being in fact equivalent to it, and therefore it is also homologous with multiplication by the imaginary unit. But there is also another homology. Let us designate the four sides of the cube parallel to the axis of rotation as A, B, C, D. Then our group of rotations will be homologous with the powers of a cyclic substitution between the four letters A, B, C, D.

Let us next introduce a new operator, namely, rotation around an axis at right angles to the first one, but always through an arc of 90 degrees. This introduces a new element into the problem, and enables us to change the cube from any one position to any other position, that is, to effect any interchange among the sides which would be consistent with their remaining sides of the same cube. Here we have a series of rotations which, in the case of the cube, are homologous with certain linear transformations which have been developed by Klein in his very beautiful book on the Icosahedron.

But it is also obvious that in introducing these rotations we are practically operating with quaternions, the operator being a unit vector. Thus we have a homology between certain forms of quaternion multiplications and linear transformations involving the imaginary unit. Moreover, since these rotations are also homologous with substitutions, performed on six symbols representing the six sides of the cube, it follows that there is also a homology between certain groups of substitutions and certain linear transformations involving two quantities, a numerator and a denominator, and quaternion multiplication by unit vectors.

I have taken a cube as the simplest illustration. Evidently we can construct a great number of groups of substitutions of the same sort between the sides of any regular solid, as Klein has done in the work I have already cited. The relation between the linear substitutions thus found and the solution of corresponding algebraic equations forms one of the most beautiful branches of our modern mathematics.

The idea of groups of operations, as I have tried to develop it, has in recent years been so extended as to cover a large part of the fields of algebra and geometry. Among the leaders in this extension has been Sophus Lie. Considered from the algebraic point of view, his idea in its simplest form may be expressed thus: We have a certain quantity, say x. We have also an operation of any sort which we may perform upon this quantity. Let this operation depend on a certain quantity, a, which necessarily enters into it. As one of the simplest possible examples we may consider the operation to be that of adding a to x. As the quantity a may take an infinity of values, it follows that there will be an infinity of operations all belonging to one class, which operations will be distinguished by the particular value of a in each case. We thus operate on x with one of these operators, and get a certain result, say x'. We operate on x' with a second operator, of the same class, and get a second result, say x''. If whatever operators we choose from the class, the result x'' could have been obtained from the original quantity x by some operation of the class, then these operations are such that the product of any two is equivalent to the performance of some one of them. Thus, by repeating them forever, we could get no results except such as could be obtained by some one operator. To illustrate by our simple example: if our operation consists in the addition of an arbitrary quantity to x, then we change x into x' by adding a certain quantity a, and x' into x'' by adding a second quantity b. The result of these two additions is the same as if we had added in the first place the quantity $a + b$. It needs hardly be said that the multiplication by x of any quantity would be another example of the same kind. The performance of any number of successive multiplications on a quantity is always equal to a single multiplication by the product of all the factors of the separate multiplications.

These operations are not confined to single quantities. We may consider the operation to be performed upon a system of quantities, which are thus transformed into an equal number of different quantities, each of these new quantities corresponding to one of the first system. If a repetition of the operation upon the second system of quantities gives rise to a third system, which could have been formed from the first system by an operation of the same class, then all these possible operations form a group.

The idea of such systems of operations is by no means new. It has always been obvious, since the general theory of algebraic operations has been studied, that any combination of the operations of addition, multiplication, and division could always be reduced to a system in which there would be only a single operation of division necessary,—just as in arithmetic a complex fraction, no matter what the order of complexity of its terms, can always be reduced to a single simple fraction, that is, to a ratio of two integers, but cannot, in general, be reduced to an integer. Abel made use of this theorem in his celebrated memoir on the impossibility of solving the general equation of the fifth degree.

Another field of mathematical thought, quite distinct from that at which we have just glanced, may be called the fairyland of geometry. To make a mathematician, we must have a higher development of his special power than falls to the lot of other men. When he enters fairyland he must, to do himself justice, take wings which will carry him far above the flights, and even above the sight, of ordinary mortals. To the most imaginative of the latter, a being enclosed in a sphere, the surface of which was absolutely impenetrable, would be so securely imprisoned that not even a spirit could escape except by being so ethereal that it could pass through the substance of the sphere. But the mathematical spirit, in four-dimensional space, could step out without even touching any part of the globe. Taking his stand at a short distance from the earth, he could with his telescope scan every particle of it, from centre to surface, without any necessity that the light should pass through any part of the substance of the earth. If a practised gymnast, he could turn a somersault and come down right side left, just as he looks to our eyes when seen by reflection in a mirror, and that without suffering any distortion or injury whatever. A straight line, or a line which to all our examination would appear straight, if followed far enough might return into itself. Space itself may have a boundary, or, rather, there may be only a certain quantity of it; go on forever, and we would find ourselves always coming back to the starting-point. These results, too, are not reached by a facetious use of words, but by rigorous geometrical demonstration.

The considerations which lead to the study of these forms of space are so simple that they can be traced without difficulty. When the youth begins the study of plane geometry, his attention is devoted entirely to figures drawn upon a plane. For him space has only two dimensions. To a given point on a straight line only one perpendicular can be drawn. By moving a line of any sort in the plane he can describe a surface, but a solid is wholly without his field. He cannot draw a line from the outside to the inside of a circle without intersecting it. On a given base only two triangles with given sides can be erected, one being on one side of the base, the other on the other. When he reaches solid geometry his conceptions are greatly extended. He can draw any number of perpendiculars to the same point of a straight line. If he has two straight lines perpendicular to each other, he can draw a third straight line which shall be perpendicular to both. A plane surface is not confined to its own plane, but can be moved up and down in such a way as to describe a solid. The characteristic of this motion is that it constantly carries every part of the plane to a position which no part occupied before.

Now it is a fundamental principle of pure science that the liberty of making hypotheses is unlimited. It is not necessary that we shall prove the hypothesis to be a reality before we are allowed to make it. It is legitimate to anticipate all the possibilities. It is, therefore, a perfectly legitimate exercise of thought to imagine what would result if we should not stop at three dimensions in geometry, but construct one for space having four. As the boy, at a certain stage in his studies, passes from two to three dimensions, so may the mathematician pass from three to four dimensions. He does indeed meet with the obstacle that he cannot draw figures in four dimensions, and his faculties are so limited that he cannot construct in his own mind an image of things as they would look in space of four dimensions. But this need not prevent his reasoning on the subject, and one of the most obvious conclusions he would reach is this: As in space of two dimensions one line can be drawn perpendicular to another at a given point, and by adding another dimension to space a third line can be drawn perpendicular to these two; so in a fourth dimension we can draw a line which shall be perpendicular to all three. True, we cannot imagine how the line would look, or where it would be placed, but this is merely because of the limitations of our faculties. As a surface describes a solid by continually leaving the space in which it lies at the moment, so a four dimensional solid will be generated by a three-dimensional one by a continuous motion which shall constantly be directed outside of this three-dimensional space in which our universe appears to exist. As the man confined in a circle can evade it by stepping over it, so the mathematician, if placed inside a sphere in four-dimensional space, would simply step over it as easily as we should over a circle drawn on the floor. Add a fourth dimension to space, and there is room for an indefinite number of universes, all alongside of each other, as there is for an indefinite number of sheets of paper when we pile them upon each other.

From the point of view of physical science, the question whether the actuality of a fourth dimension can be considered admissible is a very interesting one. All we can say is that, so far as observation goes, all legitimate conclusions seem to be against it. No induction of physical science is more universal or complete than that three conditions fix the position of a point. The phenomena of light show that no vibrations go outside of three-dimensional space, even in the luminiferous ether. If there is another universe, or a great number of other universes, outside of our own, we can only say that we have no evidence of their exerting any action upon our own. True, those who are fond of explaining anomalous occurrences, by the action of beings that we otherwise know nothing about, have here a very easy field for their imagination. The question of the sufficiency of the laws of nature to account for all phenomena is, however, too wide a one to be discussed at present.

As illustrating the limitation of our faculties in this direction, it is remarkable that we are unable to conceive of a space of two dimensions otherwise than as contained in one of three. A mere plane, with nothing on each side of it, is to us inconceivable. We are thus compelled, so far as our conceptions go, to accept three dimensions, and no more. We have in this a legitimate result of the universal experience through all generations being that of a triply extended space.

Intimately associated with this is the concept of what is sometimes called curved space. I confess that I do not like this expression, as I do not see how space itself can be regarded as curved. Geometry is not the science of space, but the science of figures in space, possessing the properties of extension and mobility which we find to be common to all material bodies. The question raised here is a very old one, and in a general way its history is familiar.

Mathematicians have often attempted to construct geometry without the use of what is commonly called the ninth axiom of Euclid, which seems to me best expressed by saying that in a plane only one line can be drawn which shall be parallel to another line in the plane in the sense of never meeting it in either direction. Yet every attempt to construct an elementary geometry without this axiom has been proved to involve a fallacy. This consideration led Lobatchewsky, and independently of him, I believe, Gauss, to inquire whether a geometry might not be constructed in which this axiom did not hold; in which, in fact, it was possible that if we had two parallel lines in a plane, one of them might turn through a very minute amount without thereby meeting the other line in either direction. The possibility of this was soon shown, and a system of geometry was thus constructed in which the sum of the angles of a plane triangle might be less than two right angles.

Afterward the opposite hypothesis was also introduced. It was found that, given two parallel lines in a plane, it might be supposed that they would ultimately meet in both directions. This hypothesis might even be made without there being more than one point of intersection, each straight line having a definite end and returning into itself. The geometry arising from these two hypotheses has been reduced to a rigorous system by Klein.

To guess the future of mathematical science would be a rash attempt. If made it might seem that, in view of the extraordinary works of the human intellect which mark our age, the safest course would be to predict great discoveries in this and all other branches of science. The question is sometimes asked whether a mathematical method may not yet be invented which shall be as great an advance on the infinitesimal calculus as the latter was on the methods of Euclid and Diophantus.

So far as solving problems which now confront us is concerned, I am not sure that the safest course would not be to answer such questions in the negative. Is it not true in physics as in mathematics that great discoveries have been made on unexpected lines, and that the problems which perplexed our ancestors now baffle our own efforts? We must also remember that the discovery of what could not be done has been an important element in progress. We are met at every step by the iron law of the conservation of energy: in every direction we see the limits of the possible. The mathematics of the twenty-first century may be very different from our own; perhaps the schoolboy will begin algebra with the theory of substitution-groups, as he might now but for inherited habits. But it does not follow that our posterity will solve many problems which we have attacked in vain, or invent an algorithm more powerful than the calculus.

RECENT RESEARCHES IN ELECTRICITY AND MAGNETISM.

Notes on Recent Researches in Electricity and Magnetism. Intended as a sequel to Professor Clerk-Maxwell's "Treatise on Electricity and Magnetism." By J. J. Thomson, M.A., F.R.S. Oxford, Clarendon Press, 1893. 8vo. pp. 586.

POINCARÉ remarks in his "Électricité et Optique": "La première fois qu'un lecteur français ouvre le livre de Maxwell, un sentiment de malaise, et souvent même de défiance se mêle d'abord à son admiration." And again he says: "Le savant anglais ne cherche pas à construire un édifice unique, définitif et bien ordonné, il semble plutôt qu'il élève un grand nombre de constructions provisoires et indépendantes, entre lesquelles les communications sont difficiles et quelquefois impossibles." The disconnected way in which Maxwell takes up one hypothesis after another, and then leaves his readers to select for themselves, is naturally abhorrent to a great mathematician accustomed to rigorous proof and logical sequence, and especially so to French mathematicians, who, since the time of Laplace, have certainly been masters of the art of producing clear, logical, and finished treatises. We venture to submit, however, that Maxwell's method finds its defence, at least in part, in the difference in the attitude of mind of a mathematician and a physicist. In mathematics, a theorem is proved rigorously, and the mind is completely satisfied, feeling that it has arrived at the *whole* truth of the theorem, whereas in all physical science one hypothesis or theory is simply *more probable* than another. Nothing in nature is subject to rigorous and ultimate proof, and theories are framed from time to time dependent upon the particular state of man's knowledge, yet often containing so many undetermined elements that several entirely different theories account equally well for the observed facts as we know them. Thus the probability of the electromagnetic theory of light is nearer unity than the emission theory of Newton, and Maxwell's theory of energy in the medium more probable than action at a distance, yet any theory which may be devised is but an approximation to the whole truth.

The mathematical theory which Maxwell built up twenty years ago, after profound study of Faraday's experimental researches, is mainly responsible for the great advance made in electricity and magnetism since that time, and his treatise continues to day the source from which the student must seek the great principles of the science

It was fortunate indeed that the Clarendon Press secured Professor Thomson's services in revising the proofs for the third edition of Maxwell, which appeared last year, for besides giving to this edition important notes and elucidations, his task determined him to publish a supplementary volume.

Himself an accomplished mathematician as well as a physicist, the author brings the resources of analysis to bear upon his researches with consummate skill, while at the same time he subordinates all to the final test of refined experiment.

The first fifty-two pages of the volume are devoted to the development of a physical method of considering the processes which occur in the electromagnetic field, based upon the properties of Faraday tubes of electrostatic induction in motion. An outline of the method appeared three years ago,* but it was not sufficiently complete to render it easily adaptable to particular cases. Maxwell's conception of displacement is too general in its character to easily yield to a mechanism which will picture to the mind the processes going on in the electric field, and this difficulty of comprehending his "displacement" was probably the chief reason why his views were so tardily accepted by physicists.

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* J. J. Thomson, Phil. Mag., March, 1891.

For this reason the author has devised a method which is mathematically equivalent to Maxwell's, but which is easily grasped by the student, and enables him to form a vivid mental picture of what is going on in any particular case under consideration.

The unit tube is one such that when it terminates on a conductor it corresponds in amount to the charge which in electrolysis we find associated with an atom of a univalent element. All space is supposed to be filled with Faraday tubes floating about, although they may not make themselves manifest on account of their want of arrangement. They exist apart from the molecules of the dielectric, have their seat in the ether, and the polarization of the particles of the dielectric is a secondary phenomenon. In the case of open tubes, these must end on matter, and since they join atoms instead of molecules, all electrification is accompanied by chemical dissociation, and the disappearance of it by chemical combination,—the tubes shortening to molecular dimensions. A Faraday tube in motion produces: (1) a magnetic force at right angles to the tube and to the direction in which it is moving; (2) a momentum at right angles to the tube and to the magnetic induction; (3) an electromotive intensity at right angles to the direction of motion and to the magnetic induction.

On this theory conduction, whether through gases, electrolytes, or metals, is essentially the same process, and is electrolytic in its character. Many examples in support of this general theory of conduction are given, and apparent exceptions to its application are explained.

Once having clearly given his method, and illustrated it by numerous particular cases, the author uses it constantly throughout the book to assist the student in rapidly getting the main features of the particular problem under consideration. Like Maxwell's treatise itself, we find the book full of suggestions. Thus on page 28 we read, " Another question which we have not touched upon, but which is very important, is whether the motion of the Faraday tubes through ether devoid of matter would produce magnetic force, or whether for this purpose it is necessary that the tubes should pass across ordinary matter as well as ether." He then suggests a form of experiment which is a modification of Professor Rowland's Berlin experiment,* for determining this.

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* *Berichte d. Berl. Acad.*, 1876, p. 211.

Chapter II is one of the most important and complete in the whole book. It is upon the passage of electricity through gases, and includes one hundred and fifty pages. There is no summary in an English text-book upon the very extensive literature on this subject; and Professor Thomson, who is one of the highest authorities on the discharge through gases, gives an admirable review of the work from the time of Coulomb to the present, and, at the close of the chapter, his theory of the electric discharge. As already stated, he believes in the electrolytic theory of discharge, and in fact his general theory of conduction makes the passage of electricity through gases, electrolytes, and even metals the same process. The essential results of Crookes, Hittorf, Goldstein, Liebig, De la Rue and Hugo Müller, and Spottiswoode and Moulton, are given and discussed; while the author's own researches on the conductivity of hot gases,[*] and on the discharge through exhausted tubes without electrodes,[†] are reproduced almost entire. The whole chapter is a splendid bibliography; and physicists at large, especially students who may undertake work in any particular phase of this important subject, are indebted to Professor Thomson for an immense service.

The mathematical part of the work begins with the chapter on conjugate functions. The method given by Maxwell for the solution of two-dimensional problems in electro-statics by means of conjugate functions is indirect, and requires the arbitrary assumption of relations between the functions and then seeking for a suitable transformation for the particular case. By the use of a general theorem in transformations given by Schwarz [‡] and Christoffel,[§] the proper transformations can be found by a direct process when the lines over which the potential is given are straight.

The theorem is as follows:

The perimeter of any polygon bounded by straight lines in a plane, which we can call the z plane, where $z = x + iy$, x and y being the Cartesian coördinates of a point in this plane, can be transformed into the axis of ξ in a plane which we shall call the t plane, where $t = \xi + i\eta$, ξ and η being the Cartesian coördinates of a point in this plane, points inside the polygon in the z plane transforming into points on one side of the axis of ξ.

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[*] *Phil. Mag.* (5), vol. 29, pp. 358–441, 1890.

[†] *Phil. Mag.*, Oct. 1891.

[‡] "Ueber einige Abbildungsaufgaben," *Crelle*, vol. 70, pp. 105, 120, 1869.

[§] Sul problema delle temperature stazionarie, *Annali di Matematica*, vol. 1, 1867.

The transformation which effects this is represented by the equation

$$\frac{dz}{dt}=C(t-t_1)^{\frac{a_1}{\pi}-1}(t-t_2)^{\frac{a_2}{\pi}-1}\dots(t-t_r)^{\frac{a_r}{\pi}-1}\dots(t-t_n)^{\frac{a_n}{\pi}-1},$$

where $a_1, a_2 \dots a_n$ are the internal angles of the polygon in the z plane; $t_1, t_2 \dots t_n$ are real quantities, and are the coordinates of points on the axis of ξ corresponding to the angular points of the polygon in the z plane.

The application of the theorem is illustrated by the solution of several cases treated by Maxwell and some interesting new ones, while the indirect method is also extended to some simple and beautiful results employing elliptic functions.

Almost the entire remainder of the book is given up to the theoretical and experimental consideration of alternating currents. The great importance which these currents have recently assumed in the industrial world for lighting, and the long-distance transmission of power, cannot fail to cause this part of the work to be thoroughly appreciated. The chapter on electrical waves and oscillations is an exhaustive treatment of the subject, and many suggestions are given, which will probably serve as starting-points for future investigations.

Alternating currents in two dimensions are first taken up, followed by the more practical case of periodic currents in cylindrical conductors treated generally and in particular cases.

Following the theoretical treatment of the subject comes a chapter on experiments on electromagnetic waves. A very complete account of Hertz's work is given, and the method of Faraday tubes applied in interpreting the results for the various positions of the resonator. An account of Lord Rayleigh's investigations on the laws according to which alternating currents distribute themselves among a network of conductors, and a discussion of the equations which hold when a dielectric is moving in a magnetic field, complete the volume.

We feel assured that Professor Thomson's supplementary volume will take its proper place beside Maxwell's great treatise in the library of every true student of electrical science.

GEORGE O. SQUIER.

FORT MCHENRY, MD, *November* 27, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, December 2, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Mr. Preston A. Lambert, Lehigh University; Mr. Edward Leroy Brown, Lehigh University. It was announced that the annual meeting would be held on Thursday afternoon, December 28, at half-past three o'clock. A committee of three, consisting of Mr. Legras, Professor Merriman, and Dr. Stabler, was elected to report at the annual meeting nominations for the officers and other members of the council for the coming year. Dr. G. W. Hill read a paper entitled "The application of analysis to polyedra."

THE annual meeting of the NEW YORK MATHEMATICAL SOCIETY was held Thursday afternoon, December 28, at half-past three o'clock. In the absence of the president and vice-president, Professor W. Woolsey Johnson occupied the chair. Professor Simon Newcomb delivered an address entitled "Modern mathematical thought." The thanks of the Society were tendered to Professor Newcomb, and he was requested to furnish a copy of his address for publication. The address appears in the present number of the BULLETIN. See p. 95.

Upon the recommendation of the council, the following persons, having been duly nominated, were elected to membership: Professor C. L. Doolittle, Lehigh University; Mr. F. W. Frankland, New York Life Insurance Company; Miss Ida Griffiths, New York. Reports were presented by the secretary and treasurer. The secretary stated that the membership of the Society was 239, of whom 44 resided in New York city and suburbs. The average attendance of members at the meetings had been 16, and 49 members had attended one or more meetings. Upon motion by the treasurer, an auditing committee was appointed to examine his accounts.

The committee on nominations, elected at the preceding meeting, reported a ticket for the officers and other members of the council for the ensuing year, and, a ballot being taken, this ticket was declared elected. The officers and council for 1894 are: president, Dr. Emory McClintock; vice-president, Dr. G. W. Hill; secretary, Dr. Thomas S. Fiske; treasurer, Mr. Harold Jacoby; librarian, Mr. Gustave Legras; committee of publication, Dr. Thomas S. Fiske and Professor Alexander Ziwet; other members of the council, Professor Henry B. Fine, Professor E. Hastings Moore, Professor Simon Newcomb, Professor Charlotte Angas Scott, Professor R. S. Woodward.

REPORT OF THE TREASURER FOR THE YEAR 1893.

Receipts.		*Expenditures.*	
Balance from 1892	$344.34	Printing Bulletin	$865.28
Net receipts from members and subscribers	1208.90	Stationery, postage, and miscellaneous	146.21
		Balance	541.75
	1553.24		1553.24

HAROLD JACOBY, *Treasurer.*

We have examined the treasurer's accounts for 1893, and have found them correct.

J. K. REES,
J. H. VAN AMRINGE,
THOMAS S. FISKE,
Auditing Committee.

WESLEYAN UNIVERSITY has purchased the elaborate collection of mathematical models made by L, Brill of Darmstadt, which formed a part of the German university exhibit at the World's Fair in Chicago.

PROFESSOR E. STUDY is conducting a course on the "Theory of algebraic forms" at Johns Hopkins University.

THE volume containing the lectures delivered by Professor Klein at Evanston after the close of the Chicago congress, and already announced as in course of publication by Macmillan & Co., is now ready.

T. S. F.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BERGBOHM (J.). Entwurf einer neuen Integralrechnung auf Grund der Potenzial-, Logarithmal- und Numeralrechnung. Heft II: Die irrationalen, exponentiellen, logarithmischen und cyclometrischen Integrale. Leipzig, (Teubner), 1893. Mk. 2.00

BERZOLAI (S.). Sull' assonometria ortogonale considerata come metodo di rappresentazione: nota. Pavia, Fusi, 1893. 4to. 24 pp. 8 plates.

BOLYAI (J.). The science absolute of space, translated by G. B. Halsted. Austin (Tex.), Halsted, 1893. [Reprinted from *Scientiæ Baccalaureus*, Vol. I., pp. 204–260.] $1.25

CAYLEY (A.). Collected mathematical papers. Vol. VI. Cambridge University Press (Macmillan), 1893. 4to. 601 pp. Boards. $5.50

DÖLP (H.). Die Determinanten, nebst Anwendungen auf die Lösung algebraischer und analytisch-geometrischer Aufgaben, elementar behandelt. 4th ed. Darmstadt, Brill, 1893. 8vo. 4 and 95 pp. Mk. 2.00

GÜNTHER (S.). Abriss der Geschichte der Mathematik und der Naturwissenschaften im Alterthum. (Appendix to **WINDELBAND** [W.], Geschichte der alten Philosophie, 2d ed., München, 1893.)

HAAG (P.). Cours de calcul différentiel et intégral. Paris, 1893. 8vo. 5 and 619 pp. Illustrated. Fr. 12.50

HADAMARD (J.). Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. Paris, 1893. 4to. Fr. 6.00

HEINITZ (G.). Elementare Berechnung der Zahl μ, welche den quadratischen Restcharacter bestimmt. (Diss.) Göttingen, Vandenhoeck und Ruprecht, 1893. 8vo. 33 pp. Mk. 1.00

HOLGATE (T. F.). On certain ruled surfaces of the fourth order. Accepted as a dissertation for the degree of doctor of philosophy at Clark University, Worcester, Mass., on the recommendation of W. E. Story. Baltimore, Friedenwald, 1893. (Reprint from the Am. Journ. of Math., Vol. 15.) 4to. 43 pp.

KLEIN (F.). Vorlesungen über höhere Geometrie. Heft II: Sommersemester 1893. Verden (Hannover), Schilling, 1893. Small 4to. 388 pp. (Autogr.) Mk. 5.00

KLEIN-SWORMINK (A. J.). De regelmatige Veelvlakken. Zutphen, 1893. 8vo. 58 pp. 1 plate. Mk. 1.50

LEVI-CIVITA (T.). Sugli infiniti ed infinitesimali attuali quali elementi analitici. Venezia, Ferrari, 1893. 8vo. 51 pp. (Atti del R. Istituto Veneto, Vol. IV, Ser. VII, 1892–93.)

LINDELÖF (L.). Sur les systèmes complets et le calcul des invariants différentiels des groupes continus finis. Paris, 1893. 4to. 62 pp. Fr. 3.50

Loud (F. H.). The circular locus. (Colorado College Studies, 1893, pp. 5–80.)

Oltramare (G.). Essai sur le calcul de généralisation. Genève, 1893. 8vo. 132 pp. (Autogr.) Fr. 6.50

Porchon (P.). Compléments d'algèbre et de géométrie (applications de l'algèbre à la géométrie, courbes usuelles). Paris, 1893. 12mo. Illustrated. Cloth. Fr. 3.00

Speckmann (G.). Beiträge zur Zahlenlehre. Oldenburg, Eschen und Fasting, 1893. 8vo. 5 and 64 pp. Mk. 2.00

Weber (E. von). Studien zur Theorie der infinitesimalen Transformation. [Diss.] München, 1893. 8vo. 69 pp.

Weierstrass (K.). Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen. Nach seinen Vorlesungen und Aufzeichnungen bearbeitet und herausgegeben von H. A. Schwarz. Abtheilung I. 2d ed. Berlin, Springer, 1893. 4to. 12 and 96 pp. Mk. 10.00

Wiener (H.). Sechs Abhandlungen über das Rechnen mit Spiegelungen, nebst Anwendungen auf die Geometrie der Bewegungen und auf die projektive Geometrie. Leipzig, Breitkopf und Härtel, 1893. 8vo. (Reprinted from Berichte der Sächs. Ges. der Wiss. 1890–1893.) Mk. 5.00

Zeuthen (H. G.). Forelæsning over Mathematikens Historie. Oldtid og Middelalder. Kjöbenhavn, 1893. 8vo. 304 pp. Mk. 6.20

II. ELEMENTARY MATHEMATICS.

Arnoux (V.). 2500 problèmes d'arithmétique et de géométrie pratique. Problèmes d'examen. Livre du maitre. Paris, 1893. 12mo. 490 pp. Illustrated. Fr. 2.50

Bardey (E.). Methodisch geordnete Aufgabensammlung, mehr als 8000 Aufgaben über alle Theile der Elementar-Arithmetik enthaltend. 19th ed. Leipzig, Teubner, 1893. 8vo. 14 and 330 pp. Mk. 2.70

Bussler (F.). Die Elemente der Mathematik für das Gymnasium bearbeitet. I: Pensum für das Untergymnasium (Quarta bis Untersecunda). II: Pensum für das Obergymnasium (Obersecunda und Prima). Dresden, Ehlermann, 1893. 8vo. 4 and 144 pp.; 4 and 212 pp. Illustrated. Boards. Mk. 3.70

Fulcheris (P.). Elementi di geometria, ad uso delle scuole tecniche e normali. Testo e tavole. 14th ed. Torino, Paravia, 1894. 16mo. 128 pp. 14 plates. Fr. 2.00

Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 40th ed. Halle, 1893. 8vo. 166 and 35 pp. Cloth. Mk. 2.50

Hauck (A. F.). Lehrbuch der Arithmetik und Algebra für Gymnasien, Real- und Handelsschulen. (In 3 Teilen.) Teil III, Abt. I. 3d ed., by J. Brunotte. Resultate. Nürnberg, Korn, 1893. 8vo. 8 and 52 pp. Mk. 1.20

Heilermann (H.) und **Diekmann** (J.). Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Theil I. 6th ed. Essen, Baedeker, 1893. 8vo. 9 and 225 pp. Mk. 2.25

Jackwitz (E.). Der Satz des Cavalieri im mathematischen Unterricht des Gymnasiums. [Progr.] Schrimm, 1893. 4to. 8 pp. 1 plate.

Jelinek (L.). Mathematische Tafeln für technische Anstalten, besonders für höhere Gewerbeschulen. (In 2 Theilen: I. Tafeln; II. Einrichtung und Gebrauch der Tafeln.) Wien, Pichler, 1893. 8vo. 4 and 223 pp. Mk. 2.40

Kühl (J. H.). Leitfaden der Arithmetik und Algebra. Für den Schul- und Selbstunterricht bearbeitet. Theil I. 2d ed. Hamburg, Kriebel, 1893. 12mo. 6 and 305 pp. Boards. Mk. 3.00

Lodi (E.). Matematica elementare, corso liceale. Vol. I: Algebra. Bologna, Gamberini e Parmeggiani, 1893. 8vo. 136 pp. Fr. 2.00

Malagodi (A.). Nozioni d'algebra elementare, con numerosi esercizi e problemi ad uso della terza classe delle scuole tecniche. Mirandola, Cagarelli, 1893. 8vo. 67 pp. Fr. 0.50

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—— Reductionstafeln $\cos^2 \alpha$ für alle Distanzen von 1-100 und alle Winkel von 1°-36° von 10′ zu 10′, dann für die Distanzen 1-10 und die Winkel 37°-46° von 10′ zu 10′ auf 2 resp. 3 Decimalen fertig berechnet. (German and Hungarian.) Vienna, 1893. 8vo. 40 pp. Mk. 1.10

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KLEIN'S EVANSTON LECTURES.

The Evanston Colloquium: Lectures on mathematics, delivered from Aug. 28 to Sept. 9, 1893, before members of the Congress of Mathematics held in connection with the World's Fair in Chicago, at Northwestern University, Evanston, Ill., by FELIX KLEIN. Reported by Alexander Ziwet. New York, Macmillan, 1894. 8vo. x and 109 pp.

THIS little volume occupies a somewhat unique position in mathematical literature. Even the *Commission permanente* would find it difficult to classify it and would have to attach a bewildering series of symbols to characterize its contents. It is stated as the object of these lectures "to pass in review some of the principal phases of the most recent development of mathematical thought in Germany"; and surely, no one could be more competent to do this than Professor Felix Klein. His intimate personal connection with this development is evidenced alike by the long array of his own works and papers, and by those of the numerous pupils and followers he has inspired. But perhaps even more than on this account is he fitted for this task by the well-known comprehensiveness of his knowledge and the breadth of view so characteristic of all his work.

In these lectures there is little strictly mathematical reasoning, but a great deal of information and expert comment on the most advanced work done in pure mathematics during the last twenty-five years. Happily this is given with such freshness and vigor of style as makes the reading a recreation. As the preface tells us, the twelve lectures here reproduced were given before twenty or more mathematicians from American colleges and universities, during the two weeks following the Chicago Congress of Mathematics last summer. Professor Klein had attended the Congress as official commissioner from Germany, and had contributed much to the success of that gathering. With the eager co-operation of a majority of the members of the Congress, he arranged the Colloquium at Evanston. This by no means easy task he undertook solely from love of his chosen science, and in the hope that thus he might contribute to the promotion of research in America in those parts of pure mathematics which at present engage the attention of the foremost European investigators. The noteworthy list of interested auditors prognosticates the realization of this hope. It is to the worker in special fields, and to the ambitious student, that these lectures will prove most useful. A complete review being in the nature of the case impossible, we shall present only briefly points of special interest.

First is discussed the geometrical work of Clebsch, it being presumed that his contributions to the theory of algebraic forms are too generally known in this country to require notice. Regarding as his most secure claim to remembrance the union that he effected between Riemann's theory of Abelian functions and the discoveries of Cayley, Sylvester, and Aronhold relating to algebraic curves, Klein states two objections to the presentation of this union given in Clebsch and Gordan's book. (1) It assumes without proof as the most general algebraic curves those that have no higher singularities than crunodes and acnodes; and (2) by employing the curve as the fundamental thing, to the exclusion of the equivalent Riemann surface, it renders obscure some points that by the other means would become self-evident, e.g., the invariant character of the deficiency-number p. The extension of the methods of the curve theory by Clebsch and his followers to the treatment of surfaces and of differential equations is emphasized as opening a rich and not too remote field of inquiry.

Two parts of Sophus Lie's closely connected researches come in for extended notice,—his sphere-geometry and his theory of "contact-transformations" as applied to linear differential equations. The former is shown to correspond to the line-geometry of Plücker, tangent spheres being correlated to intersecting lines. We quote from p. 17: "Perhaps the most striking example of the fruitfulness of this work of Lie's is his discovery that by means of this transformation the lines of curvature of a surface are transformed into asymptotic lines of the transformed surface, and *vice versa*. This appears by taking the definition given above for the lines of curvature and translating it word for word into the language of line-geometry. Two problems in the infinitesimal geometry of surfaces, that had long been regarded as entirely distinct, are thus shown to be really identical. This must certainly be regarded as one of the most elegant contributions to differential geometry made in recent times."

From most original thinkers we look for the more lucid and complete exposition of their theories in their later publications. One is therefore surprised at the following hint (p. 9): "To fully understand the mathematical genius of Sophus Lie, one must not turn to the books recently published by him in collaboration with Dr. Engel, but to his earlier memoirs, written during the first years of his scientific career. There Lie shows himself the true geometer that he is," etc. No doubt the contrary advice would be preferable for those whose bent is rather to analysis than to geometry. We venture the opinion that most students would find in Klein's own latest lecture-course (Höhere Geometrie II, lithographed, Göttingen, 1893) the best possible introduction to Lie's works.

Two lectures on recent papers of his own are full of germinal ideas, those on the real shape of algebraic curves and surfaces, and on theory of functions and geometry. The former brings to light certain problems most open to attack, and offers aid from the theory of symmetric Riemann surfaces; the latter presents in outline an ingenious application of geometric intuition to an analytical problem. So simple a thing as the structure of a multiply covered spherical triangle is made to indicate the number of zeros of a hypergeometric function between given limits.

On the subject of the validity and degree of exactness of mathematical definitions and axioms we find mainly a résumé of his paper in the *Math. Annalen*, vol. 22, with the added interest of several apt illustrations. To show the unreliability of the "naïve intuition," a curve is defined by purely geometric terms, which proves to have indeed a tangent at every point, but nowhere a determinate curvature! The conclusions reached are made a text for urging upon our schools the combination, in instruction, of the applied sciences with the abstract. "Separation could only be deplored, for it would necessarily be followed by shallowness on the side of the applied sciences, and by isolation on the part of pure mathematics."

The famous proofs by Hermite, Lindemann, and Weierstrass, of the transcendental character of the numbers e and π, previously inaccessible to most college students, have been replaced within the past year by far simpler proofs by Hilbert, Hurwitz, and Gordan; their papers are now collected in vol. 43 of the *Math. Annalen.* A lecture upon Hilbert's proof states that "The problem has thus been reduced to such simple terms that the proofs for the transcendency of e and π should henceforth be introduced into university teaching everywhere."

The views presented on the solution of higher algebraic equations will not be novel to readers of the *Ikosaeder*, but are perhaps even more sharply and forcibly stated here. In opposition to a current restriction of the "solution of an equation" to mean its solution by radicals, Klein insists that solution shall denote the reduction of an equation to one of certain definite *normal equations*. Having accomplished this for equations of degree 5, 6, and 7, he formulates the general problem as follows, first bringing into one class all equations whose Galois groups are isomorphic: "Among the problems having isomorphic groups we consider as the simplest the one that has the least number of variables, and call this the *normal problem.* This problem must be considered as solvable by series of any kind. The question is, to reduce the other isomorphic problems to the normal problem." For the general equation of the eighth degree no normal problem of lower order has yet been found.

Topics which we can only name here are ideal numbers, recent advances in hyperelliptic and Abelian functions, and most recent researches in non-Euclidean geometry. The terms *line-lattice* and *point-lattice* strike us as felicitous. We learn that a space of zero-curvature is not necessarily infinite; and that our geometrical demonstrations "have no absolute objective truth, but are true only for the present state of our knowledge." For "we can never tell whether an enlarged conception may not lead to further possibilities that would have to be taken into account. From this point of view we are led in geometry to a certain modesty, such as is always in place in the physical sciences."

The closing lecture is devoted to a description of methods adopted in the mathematical department at Göttingen and to some sound advice to students intending to go to Germany for the purpose of studying mathematics. "As regards my own higher lectures," says Prof. Klein, "I have pursued a certain plan in selecting the subjects for different years, my general aim being *to gain, in the course of time, a complete view of the whole field of modern mathematics, with particular regard to the intuitional or* (in the highest sense of the term) *geometrical standpoint.* This general tendency you will, I trust, also find expressed in this colloquium, in which I have tried to present, within certain limits, a general programme of my individual work."

The book contains as an appendix a translation (by Prof. H. W. Tyler) of Klein's sketch "The development of mathematics at the German universities," written originally for the Exposition volume "Die deutschen Universitäten." This brief historical essay serves as an introduction to the lectures in so far as it carries the history up to the year 1870, while the lectures are devoted to the last quarter of a century.

HENRY S. WHITE.

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ON THE NUMBER OF INSCRIPTIBLE REGULAR POLYGONS.

BY LEONARD E. DICKSON, B.SC.

LET us denote for brevity the phrase, a regular polygon which is geometrically inscriptible, by P. The P's are few in number compared with the non-P's. The number of P's up to 100 is 24; up to 300 is 37; up to 1000 is 52; up to 1,000,000, only 206. Indeed, a P must have for the number of its sides a *prime* number of the form $2^x + 1$, or the product of a power of 2 by any number of different primes of that form. This was proven by Gauss; and, more simply, by myself in another paper.* Further, x here must be a power of 2. For if x contains an odd factor m, such that $lm = x$, then will $2^x + 1$ be divisible by $2^l + 1$. This is seen by writing y for 2^l in $2^{lm} + 1$, which thus becomes $y^m + 1$, a number divisible by $y + 1$. But the inverse of this, that all numbers of the form $(2^{2^y} + 1)$ are prime, is not generally true, as Fermat affirmed. Euler pointed out that this rule [true for $y = 0$, 1, 2, 3, 4] fails for $y = 5$. Again, it is stated in Lucas' *Théorie des Nombres*, pages 51 and 448, that it fails for $y = 6$; but that we are still ignorant in the case of $y = 7$. Hence, the numbers $(2^{32} + 1)$ and $(2^{64} + 1)$ do not give P's; while $(2^{128} + 1)$ may or may not give a P. Thus the P's below $2^{32} + 1$ are given by 2^x times one, or 2^x times the product of two or more different ones, of the primes $2^1 + 1, 2^2 + 1, 2^4 + 1, 2^8 + 1, 2^{16} + 1$; that is, will fall under one of these 32 forms: 2^x, $3 \cdot 2^x$, $5 \cdot 2^x$, $3 \cdot 5 \cdot 2^x$, $17 \cdot 2^x$, $3 \cdot 17 \cdot 2^x$, $5 \cdot 17 \cdot 2^x$, $3 \cdot 5 \cdot 17 \cdot 2^x$, $257 \cdot 2^x$, $3 \cdot 257 \cdot 2^x$, . . . $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537 \cdot 2^x$. Further, all P's between $2^{32} + 1$ and $2^{128} + 1$ also fall under the same 32 forms; for the only *new* factors which could occur, $2^{32} + 1$ and $2^{64} + 1$, are ruled out, not being prime. We cannot proceed to P's of sides greater than $2^{128} + 1$ (which requires 39 figures to express it).

The object of this paper is to give a general expression for the number of inscriptible regular polygons between certain arbitrary limits.

Theorem I. *The number of P's below $2^x + 1$ sides, where x is less than* 32, *is* $\frac{(x-1)(x+2)}{2}$. This follows from the fact that the numbers below $2^{32} + 1$ giving P's have a definite

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* See this volume of BULLETIN, p. 20.—ED.

positional relation to the powers of 2. Writing in order a few of these ; thus,

[3], 4, [5, 6], 8, [10, 12, 15], 16, [17, 20, 24, 30], 32, [34, etc.],

we see empirically that between the successive powers of 2 there lie 1, 2, 3, 4, 5, etc., P's. This may be proven by considering special ones of the above 32 forms and then generalizing; or directly by considering the most general expression for the number of sides of a P; viz., $(2^l+1)(2^m+1)\ldots(2^n+1)\cdot 2^x$. Thus we may prove that the numbers of this general form (which includes all the above 32 forms, and hence all P's below $2^{32}+1$) fall one and one only between the successive powers of 2, the first one falling between $2^{l+m\ldots+n}$ and $2^{l+m\ldots+n+1}$. Hence between any two successive powers of 2 lies one and only one number of each of the above 32 forms, the first one of any form falling just one interval later than the first one of the form preceding it. Hence, for these successive intervals, the numbers giving P's go up by the natural series of numbers; thus, omitting the powers of 2 themselves, n such numbers lie between 2^n and 2^{n+1}. Therefore, all such numbers below 2^x (x being less than 32) are included in the sum of the natural numbers up to $(x-1)$; i.e., $\frac{x(x-1)}{2}$. But the number of P's below 2^x+1 of the form 2^x is $(x-1)$. Hence the total number of P's of sides below 2^x+1, where x is less than 32, is $(x-1)+\frac{x}{2}(x-1)=\frac{(x-1)(x+2)}{2}$.

Theorem II. *The number of P's below* 2^x+1 *sides, where* x *is greater than* 32 *but less than* 128, *is* $(32x-497)$.

The number of P's up to $2^{32}+1$ sides is 527 by our last formula. But the number of those between 2^{32} and 2^{33} sides is 31 instead of 32, as we would expect ; for $2^{32}+1$ does not give a P. Similarly, the number of P's between 2^{33} and 2^{34} is 31 instead of 33 ; for $2(2^{32}+1)$ and $3(2^{32}+1)$ do not give P's. But also $(2^{64}+1)$ is ruled out. Hence, generally, the number of P's between any two successive powers of 2 (between 2^{32} and 2^{128}) is always 31. Hence, omitting the powers of 2, the number of P's between $2^{32}+1$ and 2^x+1 is $31(x-32)$; and including the $(x-32)$ powers of 2, is $32(x-32)$. Hence the total number of P's below 2^x+1, x being between 32 and 128, is $527+32(x-32)=32x-497$.

A more elegant proof of this is as follows: Below 2^x+1 sides, the form 2^x includes $(x-1)$ P's ; the form $3\cdot 2^x$ includes $(x-1)$; the form $5\cdot 2^x$ includes $(x-2)$; $3\cdot 5\cdot 2^x$ includes $(x-3)$; $17\cdot 2^x$ includes $(x-4)$; and so on for all 32 forms. Thus the whole number of P's below 2^x+1, x being less than 128 and greater than 32, is $(x-1)+\{(x-1)+(x-2)\ldots+(x-31)\}=x-1+31(x-16)=32x-497$.

UNIVERSITY OF TEXAS, *November*, 1893.

BIBLIOGRAPHY OF MATHEMATICAL DISSERTATIONS.

Catalogue des Thèses de Sciences soutenues en France de 1810 *à* 1890 *inclusivement.* Par ALBERT MARIE, Bibliothécaire universitaire. Paris, H. Welter, 1892. 8vo. xi+224 pp.

Verzeichnis der seit 1850 *an den deutschen Universitäten erschienenen Doctor-Dissertationen und Habilitationsschriften aus der reinen und angewandten Mathematik.* Herausgegeben auf Grund des für die Universitäts-Ausstellung in Chicago erschienenen Verzeichnisses. München, 1892. 8vo.

IT is generally a matter of considerable difficulty to obtain information in regard to doctors' dissertations. They appear unannounced, at irregular intervals, in many places, and are usually the productions of authors as yet unknown. Separate publications of no great size, relating to subjects of a highly special character, they are often of but little interest except to the author. Still they are sometimes of the highest scientific importance, and the two recently published lists of the dissertations of the French and German universities will be of great value.

The contents and arrangement of the French catalogue are described in its preface, from which the following notes are extracted.

The special schools of science established in 1802 conferred no degree, and the creation of the faculties of mathematical and physical sciences in 1808 marks the commencement in France of courses in science leading to the doctorate.

The total number of dissertations at Paris from 1811 to the end of 1890 is 701. These are classified as follows: mathematics 184, physics 281, natural science 236.

Those from the Departments* number 172, classified as follows: mathematics 44, physics 61, natural science 67.

The arrangement of the list is first according to location, and then chronological. Each entry gives the name of the author; place and date of birth, and of death (if dead); official positions held; title of dissertation, publisher, date of publication, number of pages, size; register number, and date of presentation. The absence of the register number indicates, in the case of Paris only, that the dissertation was not accepted. This is not true for the Departments, since owing to imperfect registration not only the numbers but often the dates of acceptance have been omitted. The work closes with an index to authors, and a very extensive alphabetical subject-index. In the latter many of the dissertations are mentioned in several places corresponding to the principal words of their titles.

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* Besançon 8; Bordeaux 5; Caen 7; Clermont-Ferrand 1 (1858); Dijon 10; Grenoble 9; Lille 8; Lyon 14; Marseille 5; Metz 1 (1813); Montpellier 36; Nancy 9; Poitiers 2; Rennes 2; Strassburg 52; Toulouse 16.

The German list occurs in two editions. It was originally compiled under the direction of the Minister of Instruction of Prussia for the special catalogue * of the mathematical exhibit of the German universities at Chicago. The collection of dissertations of the University of Marburg was the basis of the compilation. Feeling that the list would be of interest to a larger circle of readers, the German Mathematical Association obtained permission to have it published separately. By the help of the other university libraries several titles which were omitted from the other edition were supplied.

The list gives the author's name, title of dissertation, date, and size. Those from each university are arranged together in alphabetical order. The number of doctors' dissertations since 1850 is as follows: Berlin 96; Bonn 43; Breslau 56; Erlangen 24; Freiburg i/B. 11 ; Giessen 20; Göttingen 134 ; Greifswald 15 ; Halle 78; Heidelberg 18 ; Jena 82; Kiel 19; Königsberg 31 ; Leipzig 65; Marburg 91 ; München 29; Münster 16 ; Rostock 52; Strassburg 17; Tübingen 33 ; Würzburg 7—total number 939. The list also includes 50 inaugural addresses (Habilitationsschriften).

"The list shows most clearly, by the many fields of work which the papers cover, the individual characters of the several universities and how each teacher develops in his instruction the subjects of his own investigation. Herein is to be found the fullest expression of the efficiency of the mathematical seminary." †

A comparison of the two lists shows that since 1850 the twenty-one German universities have accepted 939 dissertations upon mathematical subjects, and that from 1850 to 1890 those accepted at Paris numbered about 144, and from the sixteen Departments about 24.

A list of the small but increasing number of dissertations accepted by the American colleges and universities would be of considerable interest, as it would tend to show to what ex-

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* Special-Katalog der mathematischen Ausstellung (Gruppe X der Universitäts-Ausstellung). Bearbeitet von Dr. Walter Dyck. Berlin. 1893. 8vo. xii+116 pp.

† Preface to the Special-Katalog.

tent and in what directions the authors of these products of an "infant industry" may be expected to develop. Let us hope that some one will undertake the task.

E. M. BLAKE.

COLUMBIA COLLEGE, *January*, 1894.

THE TEACHING OF MATHEMATICS IN THE SECONDARY SCHOOLS.

EXTRACT FROM THE REPORT RENDERED TO THE NATIONAL EDUCATIONAL SOCIETY, DECEMBER 4, 1893, BY THE COMMITTEE ON SECONDARY SCHOOL STUDIES.[*]

At the meeting of the National Educational Association held in Saratoga, July 9, 1892, a committee of ten was appointed to organize a series of conferences upon the subjects and methods of instruction in the secondary schools and to prepare a report based upon the results thus obtained. The members of this committee were: Charles W. Eliot, President of Harvard University, Cambridge, Mass., *Chairman;* William T. Harris, Commissioner of Education, Washington, D. C.; James B. Angell, President of the University of Michigan, Ann Arbor, Mich.; John Tetlow, Head Master of the Girls' High School and the Girls' Latin School, Boston, Mass.; James M. Taylor, President of Vassar College, Poughkeepsie, N. Y.; Oscar D. Robinson, Principal of the High School, Albany, N. Y.; James H. Baker, President of the University of Colorado, Boulder, Colo.; Richard H. Jesse, President of the University of Missouri, Columbia, Mo.; James C. Mackenzie, Head Master of the Lawrenceville School, Lawrenceville, N. J.; Henry C. King, Professor in Oberlin College, Oberlin, Ohio.

The persons selected by them to hold the conference upon mathematical studies were the following: Professor William E. Byerly, Harvard University, Cambridge, Mass; Professor Florian Cajori, Colorado College, Colorado Springs, Colo.; Arthur H. Cutler, Principal of a Private School for Boys, New York City; Professor Henry B. Fine, College of New Jersey, Princeton, N. J.; W. A. Greeson, Principal of the High School, Grand Rapids, Mich.; Andrew Ingraham, Swain Free School, New Bedford, Mass.; Professor Simon Newcomb, Johns Hopkins University, and Washington, D. C.; Professor George D. Olds, Amherst College, Amherst, Mass.; James L. Patterson, Lawrenceville School, Lawrenceville, N. J.; Professor T. H. Safford, Williams College, Williamstown, Mass.

We reproduce below the remarks of the committee in regard to the report of the mathematical conference. The report of the committee, with the reports of the several conferences appended, may be obtained from the U. S. Bureau of Education upon application.—EDITORS.

THE form of the report of the Conference on Mathematics[†] differs somewhat from that of the other reports. This report is subdivided under five headings: 1st, General Conclusions. 2d, The Teaching of Arithmetic. 3d, The Teaching of

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[*] U. S. Bureau of Education, Circular No. 205 (1893), 8vo, 249 pp.

[†] *Ibid.*, pp. 104–116.

Concrete Geometry. 4th, The Teaching of Algebra. 5th, The Teaching of Formal or Demonstrative Geometry.

The first general conclusion of the Conference was arrived at unanimously. The Conference consisted of one government official and university professor, five professors of mathematics in as many colleges, one principal of a high school, two teachers of mathematics in endowed schools, and one proprietor of a private school for boys. The professional experience of these gentlemen and their several fields of work were various, and they came from widely separated parts of the country; yet they were unanimously of opinion "that a radical change in the teaching of arithmetic was necessary." They recommend "that the course in arithmetic be at once abridged and enriched; abridged by omitting entirely those subjects which perplex and exhaust the pupil without affording any really valuable mental discipline, and enriched by a greater number of exercises in simple calculation, and in the solution of concrete problems." They specify in detail the subjects which they think should be curtailed, or entirely omitted ; and they give in their special report on the teaching of arithmetic a full statement of the reasons on which their conclusion is based. They map out a course in arithmetic which, in their judgment, should begin about the age of six years, and be completed at about the thirteenth year of age.

The Conference next recommend that a course of instruction in concrete geometry with numerous exercises be introduced into the grammar schools; and that this instruction should, during the earlier years, be given in connection with drawing. They recommend that the study of systematic algebra should be begun at the age of fourteen; but that, in connection with the study of arithmetic, the pupils should earlier be made familiar with algebraic expressions and symbols, including the method of solving simple equations. "The Conference believe that the study of demonstrative geometry should begin at the end of the first year's study of algebra, and be carried on by the side of algebra for the next two years, occupying about two hours and a half a week." They are also of opinion "that if the introductory course in concrete geometry has been well taught, both plane and solid geometry can be mastered at this time." Most of the improvements in teaching arithmetic which the Conference suggest "can be summed up under the two heads of giving the teaching a more concrete form, and paying more attention to facility and correctness in work. The concrete system should not be confined to principles, but be extended to practical applications in measuring and in physics."

In regard to the teaching of concrete geometry, the Conference urge that while the student's geometrical education should begin in the kindergarten, or at the latest in the primary school, systematic instruction in concrete or experimental geometry should begin at about the age of ten for the average student, and should occupy about one school hour a week for at least three years. From the outset of this course the pupil should be required to express himself verbally as well as by drawing and modelling. He should learn to estimate by the eye, and to measure with some degree of accuracy, lengths, angular magnitudes, and areas; to make accurate plans from his own measurements and estimates; and to make models of simple geometrical solids. The whole work in concrete geometry will connect itself on the one side with the work in arithmetic, and on the other with elementary instruction in physics. With the study of arithmetic is therefore to be intimately associated the study of algebraic signs and forms, of concrete geometry, and of elementary physics. Here is a striking instance of the interlacing of subjects which seems so desirable to every one of the nine Conferences.

Under the head of teaching algebra, the Conference set forth in detail the method of familiarizing the pupil with the use of algebraic language during the study of arithmetic. This part of the report also deals clearly with the question of the time required for the thorough mastery of algebra through quadratic equations. The report on the teaching of demonstrative geometry is a clear and concise statement of the best method of teaching this subject. It insists on the importance of elegance and finish in geometrical demonstration, for the reason that the discipline for which geometrical demonstration is to be chiefly prized is a discipline in complete, exact, and logical statement. If slovenliness of expression, or awkwardness of form, is tolerated, this admirable discipline is lost. The Conference therefore recommend an abundance of oral exercises in geometry—for which there is no proper substitute—and the rejection of all demonstrations which are not exact and formally perfect. Indeed throughout all the teaching of mathematics the Conference deem it important that great stress be laid by the teacher on accuracy of statement and elegance of form as well as on clear and rigorous reasoning. Another very important recommendation in this part of the report is to be found in the following passage: "As soon as the student has acquired the art of rigorous demonstration, his work should cease to be merely receptive. He should begin to devise constructions and demonstrations for himself. Geometry cannot be mastered by reading the demonstrations of a text-book; and while there is no branch of elementary mathematics in which purely receptive work, if continued too long, may lose its interest more completely, there is also none in which independent work can be made more attractive and stimulating." These observations are entirely in accordance with the recent practice of some colleges in setting admission examination papers in geometry which demand of the candidates some capacity to solve new problems, or rather to make new application of familiar principles.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, January 6, at half-past three o'clock, the vice-president, Dr. G. W. Hill, in the chair. Mr. John Franklin Shields, of the Adelphi Academy, Brooklyn, having been duly nominated, and being recommended by the council, was elected a member. Dr. Blake made some general remarks in connection with the recently published concluding volume of Lie's "Theory of Transformation Groups," edited by Engel.

THE lists of substitution groups published by Professor Cole in the last volume of the BULLETIN (No. 7, pp. 184–190, No. 10, pp. 250–258) have been reprinted in the *Quarterly Journal of Mathematics*, vol. 26, pp. 372–388, forming part of another article by Professor Cole, viz. "List of the substitution groups of nine letters."

WE have to record the death, on January 1, 1894, of Heinrich R. Hertz, professor of physics in the University of Bonn, at the age of thirty-seven years.

THE professorship of mathematics at Tulane University, vacated through the death of Professor J. L. Cross, has been filled by the appointment of Dr. W. B. Smith, recently of the Missouri State University. T. S. F.

PROFESSOR FLORIAN CAJORI'S History of Mathematics has just been published by Macmillan & Co., of New York. The same house announces as in preparation a volume on Hydrostatics, by A. G. Greenhill, and a new edition of Lord Rayleigh's Theory of Sound, comprising the whole subject in two volumes; also a volume on Modern Plane Geometry, by G. Richardson and A. S. Ramsey, and one on Geometrical Conic Sections, by Charles Smith. A. Z.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Algorismus prosaycus Magistri Christani anno fere 1400 scriptus. Nunc primum edidit Dr. F. J. Studnička. Pragæ, 1893. (Sumptibus R. Soc. Scient. Boh.) 17 pp.

Brutel de la Rivière (C. J. E.). Aanhangsel tot de Handleidning bij de Beschouwing der Figuren in de Ruimte. Leiden, 1893. 8vo. 2 and 20 pp. Mk. 0.80

Byerly (W. E.). An elementary treatise on Fourier's series and spherical, cylindrical, and ellipsoidal harmonics, with applications to problems in mathematical physics. Boston, Ginn, 1893. 8vo. 8 and 287 pp. Illustrated. Cloth. $3.00

Cantor (M.). Vorlesungen über Geschichte der Mathematik. (In 3 Bänden.) Vol. 1: Von den ältesten Zeiten bis zum Jahre 1200 n. Chr. 2te, umgearbeitete Auflage. Leipzig, Teubner, 1894. 8vo. Illustrated. Mk. 20.00

Collette (L.). Exercices de calcul intégral. Liége, 1893. 8vo. Mk. 2.80

Deter (C. J.). Repetitorium der Differential- und Integralrechnung. 3te Auflage. Berlin, Rockenstein, 1893. 8vo. 119 pp. Illustrated. Mk. 2.00

Durège (H.). Elemente der Theorie der Funktionen einer complexen veränderlichen Grösse. Mit besonderer Berücksichtigung der Schöpfungen Riemann's. 4te Auflage. Leipzig, Teubner, 1894. 8vo. 10 and 300 pp. Mk. 6.80

Gauss (C. F.). De integratione formulæ differentialis $(1 + n \cos \phi)^{\nu}.d\phi$. Herausgegeben von E. Schering. Göttingen (Nachr. Ges. Wiss.), 1893. 8vo. 30 pp. Mk. 1.80

Klein (F.). The Evanston Colloquium. Lectures on mathematics, delivered from Aug. 28 to Sept. 9, 1893, before members of the Congress of Mathematics held in connection with the World's Fair in Chicago, at Northwestern University, Evanston, Ill. Reported by A. Ziwet. Published for H. S. White and A. Ziwet. New York, Macmillan, 1894. 8vo. 10 and 109 pp. Cloth. $1.50

Le Vavasseur (R.). Sur le système d'équations aux dérivées partielles simultanées aux quelles satisfait la série hypergéométrique à deux variables $F(\alpha, \beta, \beta', \gamma; x, y)$. Paris, 1893. 4to. 211 pp. Illustrated. Mk. 6.00

MacMahon (P. A.). Memoir on the theory of the composition of numbers. London (Phil. Trans.), 1893. 4to. 67 pp. 8s.

Molenbroek (P.). Anwendung der Quaternionen auf die Geometrie. Leiden, 1893. 8vo. 10 and 257 pp. Mk. 7.00

Stringham (I.). Uniplanar algebra, being Part I of a propædeutic to the higher mathematical analysis. San Francisco, Berkeley Press, 1893. 8vo. 14 and 141 pp. Cloth. $1.25

II. ELEMENTARY MATHEMATICS.

Ambrosino (V.). Elementi di trigonometria sferica, ad uso delle scuole secondarie. Messina, Trimarchi, 1893. 8vo. 50 pp., 1 plate.

Briggs (W.) and Bryan (G. R.). Elements of co-ordinate geometry. Part I: the equations and properties of the right line and circle. 2d ed. London, Clive, 1893. (Univ. Corr. Coll. Tutorial Series.) 8vo. 216 pp. 3s. 6d.

Briggs (W.) and Edmondson (T. W.). Mensuration of the simpler figures, including the elements of the geometry of the rectilinear solids. London, 1893. 8vo. 170 pp. Illustrated. Cloth. 3s. 6d.

Dupuis (N. F.). Elements of synthetic solid geometry. New York, Macmillan, 1894. 12mo. 239 pp. Cloth. $1.60

Gremigni (M.). Ancora a proposito del postulato dell' equivalenza e di altre questioni geometriche: seconda risposta al prof. Lazzeri. Firenze, Fiorentino, 1893. 8vo. 8 pp.

Hall (H. S.) and Knight (S. R.). Elementary trigonometry. London and New York, Macmillan, 1893. 12mo. 14 and 356 pp. Cloth. $1.10

Jones (G. W.). Logarithmic tables. 5th edition. London, Macmillan; and Ithaca, Jones; 1894. 8vo. $0.75

Lodi (E.). Matematica elementare, corso liceale. Vol. III: Trigonometria. Bologna, Gamberini e Parmeggiani, 1893. 8vo. 59 pp. Illustrated. Fr. 1.00

Močnik's geometrische Formenlehre für die erste Classe der Realschulen. Bearbeitet von F. Maurer. 2te Auflage. Prag, Tempsky, 1893. 8vo. 60 pp. Illustrated. Mk. 0.90

Pincherle (S.). Algebra elementare. 5ta edizione. Milano, Hoepli, 1894. 16mo. 8 and 210 pp. (Manuali Hoepli.)

Schlotterbeck (B.). Leichte algebraische Aufgaben für das Kopfrechnen. Mit einem Anhang: Ganz leichte algebraische Aufgaben zum Kopfrechnen für die Unterstufe. Leipzig, Peter, 1893. 8vo. 56 pp. Mk. 0.60

Teegan (J. H.). Elementary and intermediate algebra. 6th ed. Dublin, Sullivan, 1893. 8vo. 450 pp. 3s. 6d.

III. APPLIED MATHEMATICS.

Ball (R. S.). In the high heavens. (Movements of the solar system; physical condition of other worlds; the great eclipse of 1893; the fifth satellite of Jupiter; the origin of meteorites; etc.) London, 1893. 8vo. Cloth. 7s. 6d.

——— The story of the sun. With 11 full-page plates and numerous illustrations. London, Cassell, 1893. 8vo. 382 pp. 21s.

Brunswig (H.). Tabellen zur Bestimmung der Breite. Hamburg, 1893. Oblong. 11 and 99 pp. Cloth. Mk. 3.00

Caspari (E.). Untersuchungen über Chronometer und nautische Instrumente. Studie über den Mechanismus und den Gang der Chronometer. Übersetzt von E. Gohlke. Bautzen, 1893. 8vo. 4 and 294 pp. 2 plates. Mk. 8.00

Clerke (A. M.). The planet Venus. London, 1893. 8vo. 58 pp. Cloth. 1s.

Cox (F. P.). Continuous-current dynamos and motors: their theory, design, and testing: an elementary treatise for students. New York, Johnston, 1893. 12mo. 271 pp. Illustrated. Cloth. $2.00

Delbos (L.). L'Astronomie aux Indes Orientales. Paris, 1893. 8vo. 32 pp. Illustrated. Mk. 1.50

De-Marchi (L.). Sulla teoria dei cicloni. Milano, 1893. 4to. 15 plates. Mk. 6.50

Faber (F.). Darstellende Geometrie mit Einschluss der Perspektive, insbesondere zum Gebrauche an Fortbildungs- und Baugewerkschulen, sowie zum Selbstunterricht. Nach dem Tode des Verfassers herausgegeben von O. Schmidt. 2 Teile. Dresden, Kühtmann, 1893. 8vo. 12 and 129 pp. 41 plates. Mk. 8.00

Guyon (E.) et **Willotte** (H.). Cours élémentaire d'astronomie. Paris, 1893. 8vo. 378 pp. 2 plates. Illustrated. Fr. 10.50

Jelinek (L.). Mathematische Tafeln für technische Anstalten. 2 Theile. Wien, 1893. 8vo. 224 pp. Mk. 2.40

Jessop (C. M.). The elements of applied mathematics, including kinetics, statics, and hydrostatics. New York, Macmillan, 1894. (Cambridge math. series.) 12mo. 110 pp. Cloth. $1.25

Lauenstein (N.). Die graphische Statik. Elementares Lehrbuch für technische Unterrichtsanstalten und zum Gebrauch in der Praxis. 2te Auflage. Stuttgart, Cotta, 1893. 8vo. 6 and 164 pp. Illustrated. Mk. 4.00

Naumann (A.). Technisch-thermochemische Berechnungen zur Heizung insbesondere mit gasförmigen Brennstoffen. Aufgaben mit ausführlichen Lösungen als Leitfaden für Praktiker und zur Übung für Studirende. Braunschweig, Vieweg, 1893. 4to. 9 and 113 pp. Mk. 6.00

Thannabaur (J.). Berechnung von Renten und Lebens-Versicherungen, an der Hand von Beispielen erläutert. Wien, Graeser, 1893. 8vo. 8 and 135 pp. Mk. 3.00

Turner (J. H. T.) and **Brightmore** (A. W.). The principles of waterworks engineering. London, Spon, 1893. 8vo. 444 pp. 25s.

Voyer (J.). Théorie élémentaire des courants alternatifs. Paris, 1893. 8vo. 88 pp. Illustrated. Mk. 1.80

Weekes (R. W.). The design of alternate current transformers. London, 1893. 8vo. 96 pp. Illustrated. Cloth. 2s.

Wells (S. H.). Engine drawing and design: a practical manual for engineering students. (In 2 vols.) Vol. I: Geometry, practical, plane, and solid. Vol. II: Machine and engine drawing and design. Philadelphia, Lippincott, 1893. 12mo. Illustrated. Cloth. $2.50

Wydts (A.). La machine électrique; sa théorie, calcul de ses éléments de construction. Paris, 1893. 8vo. Illustrated. Fr. 4.00

JORDAN'S COURS D'ANALYSE.

Cours d'Analyse de l'École Polytechnique. Deuxième édition, entièrement refondue. By C. JORDAN. Vol. I. Paris, Gauthier-Villars, 1893. 8vo. VI + 612 pp.

M. JORDAN'S Cours d'Analyse ranked as a classic from the time of its publication; and deservedly so, for it had all the excellencies of style and treatment that we have come to expect in the publications of the leading French mathematicians. The first two volumes were, in essence, a reproduction of lectures that had been given at the École Normale, and were intended to serve as an introduction to the infinitesimal calculus and to the theory of functions of a complex variable. To avoid daunting his readers M. Jordan was compelled to pass somewhat lightly over the discussion of first principles, but as far as was compatible with elementary instruction his explanations of fundamental theorems were logically exact. In the first volume of the new edition of his Cours d'Analyse, M. Jordan has introduced much new matter, and has made many changes in arrangement; for instance, he now discusses the simpler functions of the complex variable and the branches of an algebraic function as early as the second chapter, and he has incorporated in the main text the greater portion of the long note at the end of the third volume of the first edition. But more important than any alterations in detail, however extensive they may be, is the fact that the book in its present shape appeals to a new class of readers. It is intended for those that are familiar with the ordinary processes of the differential calculus, but unfamiliar with modern researches on the underlying ground-principles. Professor Klein has said very justly*: "The second edition of the Cours d'Analyse of Camille Jordan may be regarded as an example of extreme refinement in laying the foundations of the infinitesimal calculus. To place a work of this character in the hands of a beginner must necessarily have the effect that at the beginning a large part of the subject will remain unintelligible, and that, at a later stage, the student will not have gained the power of making use of the principles in the simple cases occurring in the applied sciences," but he adds, "on the other hand it is a matter of course that for more advanced students, especially for professional mathematicians, the study of works like Jordan's is indispensable."

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* The Evanston Colloquium: Lectures on mathematics delivered at Northwestern University by Felix Klein, and reported by A. Ziwet, page 49. New York, Macmillan, 1894.

The present volume contains five chapters which treat of real variables, complex variables, series, applications of Taylor's series, and plane algebraic curves.

The earlier pages of the first chapter are concerned with irrational numbers, limits, assemblages, restricted functions, continuous functions, and functions with restricted variation. When the corresponding ideas have been fully explained, the author proceeds to establish many of the usual theorems of the differential calculus on the lines marked out in the first fifty pages. As we have said above, M. Jordan's object is to explain the first principles of the subject with all possible rigor and generality. To effect his purpose he begins with the definition of the irrational number and thus paves the way for the discussion of limits, assemblages, functions, and integration regarded as a summation. Metaphysical difficulties arise when an analysis based on number-concepts is bound up with a geometry based on concepts of continuous magnitude; but it is possible to evade these difficulties. Wherever an appeal is ordinarily made to geometric experience, recourse may be had to a definition; a familiar example is that of the irrational number. The number $\sqrt{2}$ was unquestionably suggested by geometric experience, but it is possible to frame such a definition of irrational numbers in general as will permit $\sqrt{2}$ to find its place as a formal number in the number-system. When sufficient definitions have been granted as a foundation it becomes possible to build a superstructure that is logically unassailable. Such is the plan followed by many writers on real functions of real variables. This avoidance of direct reference to experience gives the subject a purely formal aspect, and the value of a subject so treated must be gauged largely by the correspondence of its definitions with the known facts of consciousness. But the moment such a test is applied we are again face to face with the metaphysical difficulties. This in nowise detracts from the value of such works as the one that we are now considering. It is of high importance that we should be able to say in the Calculus, as in Euclidean and non-Euclidean geometry, that from certain initial data all other theorems and consequences can be deduced by processes of pure reasoning.

The theory of assemblages is an essential element in the working out of M. Jordan's scheme. An assemblage E is said to be perfect when it *contains* its first derived assemblage E', whereas Cantor regards E as perfect only when it *coincides with* E'. According to M. Jordan's definition a perfect assemblage may contain isolated points. Suppose, for simplicity, that E is of one dimension and that the points of E

are strewn along a finite segment AB of a straight line; then the remaining points of the segment constitute a complementary assemblage E_1. Points belonging at once to E, E_1', or to E_1, E', where E', E_1' are the first derived assemblages from E, E_1, are called frontier points of E. All other points are said to be interior or exterior to E. It is possible that E may have no interior or exterior points, but it must have frontier points and these frontier points must form a perfect assemblage. A simple example is offered by the assemblage of rational numbers on AB; here all the points of AB are frontier points. An assemblage of points $(a, b, c, \ldots)$ in any number of dimensions is said to be restricted when the absolute values of the numbers a, b, c, ... admit an upper limit μ; and we have the well-known theorem that a restricted assemblage composed of infinitely many points has at least one limiting point. The separation (*écart*) of two points $(a, b, c, \ldots)$ and $(a', b', c', \ldots)$ being understood to mean the value of $|a - a'| + |b - b'| + |c - c'| + \ldots$, two assemblages E, F of the same character are said to be separated when the separations of points p of E from points q of F admit a lower limit Δ greater than 0. When the assemblages are restricted and perfect this lower limit is attained; that is, there exist points p, q with a separation exactly equal to Δ.

An assemblage that is perfect and restricted is said to be of a single piece (*d'un seul tenant*) when it cannot be resolved into several separate assemblages. The necessary and sufficient condition that E shall be of a single piece is that between any two points p, p' whatsoever of E it is always possible, whatever be the value of the positive number ϵ, to interpolate a chain of intermediate points that belong to E and are such that the separations of consecutive points are all less than ϵ. Such an assemblage E coincides with E'; for were E to contain isolated points, it would clearly be possible to find a value ϵ sufficiently small to break the chain from an isolated point to a second point of E. We have reproduced this theorem because it serves to illustrate the importance of having definite images before the mind's eye when dealing with abstract propositions of the kind just stated. The use of the word chain might suggest that an assemblage of a single piece is merely an analytic mode of expressing the geometric notion of a connected region, that is, a region in which it is possible to pass from any one point to any other along a continuous curve that lies wholly in the region; but to grasp the full significance of the theorem attention must be paid to the fact that E is *perfect*. To take a definite example, let all the points (a, b) whose coördinates are algebraic numbers be excluded from the square whose corners are $(\pm 1, \pm 1)$; it has been proved by Cantor[*] that any two points of the remaining assemblage can be connected by a continuous curve all of whose points belong to E. This assemblage of points has therefore the property that any one of its points can be chained to any other of its points in the manner described above, but the assemblage is not of a single piece; for each of the excluded points is a limiting point of E, and therefore E is not perfect. In the case of one dimension, an assemblage of a single piece that contains a and b contains every number between a and b. In passing it may be remarked that this result of Cantor's throws light on a theorem to which Professor Klein called attention in his Evanston lectures, viz., that with the exception of the point ($x = 0$, $y = 1$) the exponential curve $y = e^x$ has no algebraic point.

The interpretation of $\int f(x)d\sigma$, where $d\sigma$ is the element of integration, as the limit of an infinite sum, permitted Riemann to extend the meaning of integration to many cases where f is discontinuous within the region of integration. This work of Riemann's was generalized and completed by Darboux when he showed that for a restricted function the two sums $\Sigma M d\sigma$, $\Sigma m d\sigma$, where M and m stand for the upper and lower limits of f in $d\sigma$, have always perfectly determinate limits (the integral by excess and the integral by defect). A function may be treated as integrable when these limits coincide. Investigations have been carried out along these lines; of these the following theorem[†] may serve as a sample. Let the n integrable functions $\phi_\kappa(x)$, where $\kappa = 1, 2, \ldots, n$, lie between α_κ and β_κ inclusive when x lies between a and b inclusive, and let $F(x_1, x_2, \ldots, x_n)$ be continuous for values x_κ between α_κ and β_κ inclusive, then $F(\phi_1 x, \phi_2 x, \ldots, \phi_n x)$ is integrable in the interval a to b.

So far as concerns the function under the sign of integration these theorems of Riemann and Darboux are satisfactory, but they are not equally clear and precise as regards the field of integration. M. Jordan gives to the field all its generality and shows that it may be regarded as having an interior and also an exterior extent. Taking a region of two dimensions in the finite part of the plane of x, y, the recognized process for finding the area is to divide the region by parallels to the axes into a network of rectangles; some of these contain only interior points, others contain boundary points. The limit to which the sum of the former rectangles tends is the area of the region, while the latter sum tends to zero. This analysis covers all cases of common occurrence. Using a different form of language, M. Jordan divides a restricted two-dimensional assemblage of points by means of a network of squares; the points of some of these squares may be all interior to E, the corresponding squares will then be called interior squares. All the remaining squares are frontier squares. M. Jordan shows that by considering the sums of interior squares for all possible modes of division we must arrive at an upper limit called the interior extent of E, and that by considering the sums of interior and frontier squares we must similariy arrive at a lower limit called the exterior extent. It is only when the two numbers thus defined coincide that the assemblage E can be said to have a measurable extent. Although the division is made primarily by means of squares, it is shown afterwards that the same results are attained when the square elements are replaced by other elementary regions that are "quarrable." The following is a simple example of a non-measurable assemblage. Let E consist of points (x, y) for which $0 \leqq y \leqq 1$; and let x go from 0 to 1 inclusive when y is rational, from 0 to -1 inclusive when y is irrational. Here the interior extent vanishes, while the exterior extent differs from 0.

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[*] G. Cantor: "Sur les ensembles infinis et linéaires de points." *Acta Mathematica*, vol. 2, p. 368.

[†] Du Bois-Reymond: "Ein allgemeiner Satz über die Integrirbarkeit von Functionen integrirbarer Functionen." *Math. Annalen*, vol. 20, p. 122.

Sections VI and VII of chapter I are concerned with derivatives and integrals of functions of a single variable, partial derivatives and total derivatives; section IX treats of continuous lines and rectifiable lines. The discussion of what is meant by the length of a curve is very thorough, and much of the analysis is not to be found in any other text-book. The same remark applies to section XII on change of variables in definite integrals. In this section M. Jordan, to avoid impairing the generality of his work, makes no attempt to reduce the multiple integral to a succession of simple integrals, for this reduction has only been established for integrals, properly so called, taken over measurable fields; in his method all the variables are changed simultaneously.

The second chapter gives an account of complex variables and contains proofs of Cauchy's two cardinal theorems in integration.

The first of these two theorems is stated in the following form: "Let C be a closed contour which is continuous and has no multiple point, and further let C be such that all points non-exterior to C are interior to the domain E; then the integral $\int f(z)dz$ taken over any closed rectifiable line K interior to C will be identically zero." Here $f(z)$ denotes a function of z which is synectic within the domain E. Several pages are devoted to rational functions and algebraic functions; and these are followed by an exposition of the principal properties of the elementary transcendents.

Chapter III opens with proofs of Taylor's theorem for real functions of one and several real variables, and contains proofs of the corresponding theorems for functions of a complex variable. The third section relates to infinite series and products with numerical terms and gives the usual rules and theorems relating to the corresponding kinds of convergence; this leads on to series of functions and to a careful explanation of the method of continuation of integral series. On page 342 we have a statement of Weierstrass's important theorem—"If $S(u, z_1, z_2, \dots, z_n)$ be an element of an analytic function with $n + 1$ variables, such that $S(u, 0, 0, \dots, 0) = 0$ has m zero roots, then the equation $S(u, z_1, z_2, \dots, z_n) = 0$ admits, when $z_1, z_2, \dots, z_n$ are infinitely small, m infinitely small roots." The proof of this general theorem is not given, but it is used in the special case $m = 1$ as a basis for the theory of algebraic functions of one variable. When completed by the addition of some auxiliary theorems it permits one to assert with safety that if n branches of u derived from the algebraic equation $S(u, z) = 0$ vanish at $z = 0$, then each of these n branches can be represented in the neighborhood of $z = 0$ by a convergent series

$$Mz^{\mu} + M z^{\mu_1} + \dots,$$

where the μ's are positive increasing fractions such that the least common multiple m of their denominators is equal to or less than n. Space is found in this chapter for many special examples of the general theory of series; the ordinary trigonometric expansions are worked out and a short account is given of the hypergeometric series and of the gamma-function. The question of maxima and minima of functions of two variables is examined with care, and the discussion of the doubtful cases leaves few outstanding difficulties. This part of the chapter will repay careful study.

Little need be said of chapter IV, as the text is substantially the same as that of the first edition. As the title of the chapter indicates, this portion of the book indicates the rôle played by the differential calculus in the proof of geometric theorems. The number of illustrations is large and varied, and the proofs are always elegant. An especially interesting part is that which relates to ruled surfaces, congruencies and complexes.

The last chapter, on algebraic plane curves, will be of great assistance in clearing up difficulties that arise in the study of Cremona transformations. It begins with an explanation of homogeneous coördinates and homographic transformations and introduces the discriminant, Hessian and polars of a ternary form $f(x, y, z)$. The second section treats of cycles, and shows the bearing of this method on the problems of finding the numbers of intersections of two curves $f = 0$, $F = 0$ at a point that is multiple on both, and of finding the reduction in the class of $f = 0$ produced by the presence of this point. Here an opportunity is presented for defining Halphen's *characteristic exponents*. It is proved (i.) that a point p of a curve, with its associated cycles $C_1, C_2, \ldots$ of orders $r_1, r_2, \ldots$, becomes after a homographic transformation a point p' on the new curve, with the same number of associated cycles $C_1', C_2', \ldots$ of the same orders as before; (ii.) that when some of the first cycles have coincident tangents, the same is the case with the transformed cycles; and (iii.) that the characteristic numbers are unaffected. After homographic come quadratic transformations; the effect of such transformations upon cycles variously placed in regard to the triangle of reference is discussed; and a proof is given that it is possible to change any curve by a number of quadratic transformations into another curve whose cycles are of the first order, and whose tangents at the multiple points are distinct. The volume ends with a section on birational transformations of a curve, in which use is made of a particular case of Abel's theorem.

The need for a book of this kind on the principles of the differential and integral calculus has been apparent for some time. It was scarcely to be expected that a mathematician of the front rank would be able to spare the necessary time from his own researches for such an undertaking, but as usual the unexpected has happened. That the Cours d'Analyse in its new form is by no means easy to read is due to the untractable nature of the material, not to any lack of skill in the author; moreover the reader will find that the difficulties that present themselves are far from being insuperable. We hope that this new edition will meet with a recognition commensurate with its deserts, and that it will do much to increase the educational value of the differential and integral calculus as instruments of logical discipline.

J. HARKNESS.

BRYN MAWR, *February* 10, 1894.

THREE NOTES ON PERMUTATIONS.

BY PROFESSOR F. MORLEY.

1. *A plea for the chess-board in teaching determinants.*

Not a few writers on determinants, whether in works devoted to the subject or in an incidental treatment, dispose of the "rule of signs" with such a statement as: "The sign + or — is to be prefixed to a term according as it can be derived from the leading term by an even or odd number of interchanges of suffixes." It will, I think, be conceded that the rule, so stated, appears very arbitrary to a class; and I suppose that many teachers, like myself, have experienced "a certain gaspingness" in laying it down. Of course the difficulty disappears when the characters of even and odd permutations have been first explained, as is done, for example, by R. F. Scott in his treatise. The chief question for the teacher then is: which rule for determining the nature of a permutation shall I recommend—Cramer's or Cauchy's? And the question seems answered by the terms in which Jacobi (*Crelle*, vol. 22, p. 287) refers to these rules after proving them; he refers to Cramer's rule as "quam regulam," and to Cauchy's as "hanc pulchram regulam." See also the remarks of Muir, History of the Theory of Determinants, vol. 1, p. 247.

As the basis of this first note, Cauchy's writings, in particular the memoir which appeared in the *Comptes Rendus*, vol. 12, March 1841 (Works, series 1, vol. 6, p. 87; summarized in Muir's History, vol. 1, p. 234), must be especially referred to. I do not think that it has been sufficiently pointed out how to render the ideas of his abstract presentation intelligible to the average beginner, and I venture, in this first note, to sketch what seems to me the best way of presenting the whole matter to a class, namely, by fusing the idea of a term of a determinant with that of a permutation. I omit a proof which will be found in the section on substitutions in Serret's Algèbre Supérieure, or in Muir; the changes to be made being merely verbal.

We are not concerned with the size of the constituents, but only with certain elementary questions of permutations.* Only, instead of regarding the things to be permuted as restricted to a line, we regard them as material "men" actually placed on a material chess-board, in the first or standard case all on a diagonal.* We then make "moves"; regarding any two men as at opposite corners of a rectangle whose sides are parallel to those of the board, the move displaces these men to the two other corners of their rectangle.†

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* It is perhaps inadvisable to say much about substitutions in this connexion.

We can suppose the moves made always along columns; then if we look along the columns we see the men always in their original order, but if we "give a side-glance" we see the various permutations.

In the standard case every man is on the diagonal selected or is "home." A man so placed is a "cycle of the first order" or say a monad. The first move puts *a* in *b*'s row—*a* and *b* being any two men—and *b* in *a*'s. The two men, so placed, are a "cycle of the second order" or say a duad. Next move *b* and *c*; now *a* is in *b*'s row, *b* in *c*'s, *c* in *a*'s; that is, we have a cyclic interchange. The resulting position of the three men is a triad. And so on, taking (to begin with) always one of the men already moved and a new monad, in order to show the formation of cycles of the order x, say of x-ads. When the idea is grasped, we have then to introduce the cardinal proposition: a move, when the two men are of different cycles, welds the two cycles into one; and a move, when the men are of the same cycle, splits the cycle into two.‡

Now let n be the whole number of men, n_x the number of x-ads, so that in any position

$$n_1 + 2n_2 + 3n_3 + \ldots = n,$$

or

$$\Sigma x n_x = n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

In the standard position we have n monads, and it follows at once from the cardinal proposition that m, the number of moves from the standard position to any other assigned one, is congruent (mod 2) to the difference of their numbers of cycles, that is,

$$n - \Sigma n_x \equiv m, \text{ mod } 2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Thus the number of moves is either essentially odd or essentially even according to the nature of the position. When m is even, we call the position or permutation an even one;

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* The two-dimensional idea in permutations was used by Rothe in 1800. See Muir, History, vol. 1, p. 60.

† The move is of course an "interchange of suffixes" or a "transposition."

‡ Serret, vol. 2, p. 274; Muir's History, vol. 1, p. 262; see also Johnson, "On the fifteen puzzle," *American Journal*, vol. 2.

when m is odd, the position is called odd. The "rule of signs" is that the sign + or — is to be prefixed to a term according as it arises from an even position or from an odd one.

From (1) and (2) it follows that

$$n_2 + 2n_3 + 3n_4 + \ldots \equiv m;$$

in fact it requires only $x - 1$ moves to bring an x-ad home, so that the left side is the minimum of moves necessary to bring the assigned position home.

Again, the last congruence shows that

$$n_2 + n_4 + n_6 + \ldots \equiv m. \quad . \quad . \quad . \quad . \quad (3)^*.$$

2. *A special rule of signs.*

A glance at the determinant itself tells us the number of monads and duads in any selected position; and, when $n < 6$, the nature of the cycles is evident merely from the number of monads and duads. For example, when $n = 5$, we have the scheme of solutions of (1),

n_1	n_2	n_3	n_4	n_5
5	0	0	0	0
3	1	0	0	0
2	0	1	0	0
1	0	0	1	0
1	2	0	0	0
0	0	0	0	1
0	1	1	0	0

in which the first two columns determine the rest. Thus when the position is symmetrical[†] about the diagonal, so that it contains merely monads and duads,

$$\Sigma n_x = n_1 + n_2,$$

but in other cases

$$\Sigma n_x = n_1 + n_2 + 1.$$

Hence, when $n < 6$, we have from (2) the following rule: Understand by a "symmetry" either a monad (constituent on the leading diagonal) or a duad (two constituents symmetrical with respect to that diagonal). Glance along the dia-

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[*] Lucas, Récréations mathématiques, vol. 1, p. 204.

[†] The symmetrical position is Rothe's "self-conjugate permutation": positions which are reflexions of one another as to the diagonal are Rothe's "conjugate" and Jacobi's "reciprocal" permutations. Muir, pp. 59 and 245.

gonal and note the number of symmetries; only if the term consist altogether of symmetries, one of them is *forced* and is not to be counted. Let s be the number of unforced symmetries, n the number of constituents; then the term is to be taken positively or negatively according as $n - s$ is odd or even. The rule fails when $n = 6$, owing to the fact that we cannot infer, merely from the absence of symmetries, whether a position consists of two duads or a single hexad. But it does not seem worth while to complicate the rule to meet cases of comparatively infrequent occurrence, or to raise the further question whether the rule then affords any improvement on the direct application of (3).

3. *The enumeration of positions.*

When a solution of (1) has been selected, we may say that we have all positions of a given kind, and the next question is: how many positions are there of a given kind? This question affords a good exercise in combinations, but it is enough to refer to Muir (vol. 1, p. 238), Serret (vol. 2, p. 259), Netto (p. 22 of Cole's translation), or to the original sources, Cauchy (*op. cit.*) and Jacobi (*Crelle*, vol. 22, p. 372). The answer is

$$n\,!/n_1!\ 2^{n_2}.\ n_2!\ 3^{n_3}.\ n_3!\ \ldots,$$

and by summing this for all solutions of (1), that is, for all unrestricted partitions of n, we have the total number of positions, which is $n\,!$. Hence

$$\Sigma 1/n_1!\ 2^{n_2}.\ n_2!\ 3^{n_3}.\ n_3!\ \ldots = 1. \quad . \quad . \quad . \quad (4)$$

It may be remarked that Jacobi published this formula in the same year (1841) as Cauchy, in the paper just cited, at the end of the three memoirs on determinants. He gives two proofs, of which the second is combinational; from the first I wish to draw an inference or two.

We have

$$1/(1 - t) = \exp\{-\log(1 - t)\}$$

$$= \exp(t + t^2/2 + t^3/3 + \ldots).$$

Jacobi now expands the exponential and uses the polynomial theorem.*

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* Again a coincidence may be noticed: Cauchy employs a similar device, though not in immediate connexion with formula (4), in the memoir which immediately followed the one already cited; see his works, series 1, vol. 6, in particular p. 112.

We may instead write

$$1/(1-t) = \exp t \,.\, \exp t^2/2 \,.\, \exp t^3/3 \ldots$$

$$\left.\begin{array}{llllllll}
= 1 + t & + \frac{t^2}{2!} & + \frac{t^3}{3!} & + \frac{t^4}{4!} & + \frac{t^5}{5!} & + \frac{t^6}{6!} & + \ldots \\
\times 1 & + \frac{t^2}{2} & & + \frac{t^4}{2!\,2^2} & & + \frac{t^6}{3!\,2^3} & + \ldots \\
\times 1 & & + \frac{t^3}{3} & & & + \frac{t^6}{2!\,3^2} & + \ldots \\
\times 1 & & & + \frac{t^4}{4} & & & + \ldots \\
\times & . \quad . & . \quad . & . \quad . & . \quad . & . \quad . & .
\end{array}\right\} \quad . \quad . \quad (5)$$

whence the coefficient of t^n in the product of the power-series = the coefficient of t^n in $(1-t)^{-1}$, $= 1$.

Now it seems to me worthy of notice that not merely does the coefficient of $t^n/n!$ in the above product give the whole number of positions in a determinant, but each term of the coefficient is the number of positions of a given kind. If, for example, we wish for 5 monads we must take the term $t^5/5!$ from the expansion of $\exp t$; if 3 duads, the term $t^6/3!\,2^3$ from $\exp(t^2/2)$; and so on.

Therefore the number of positions with no x-ad is found by excluding the series for $\exp(t^x/x)$; thus on the one hand it is $n!\,\Sigma'$ (where Σ' is the sum in (4), with the restriction that in the summation n_x is not to appear), and on the other hand it is the coefficient of $t^n/n!$ in $\exp(-t^x/x)/(1-t)$.

Thus, when $x = 1$, the number of positions with no monad is the coefficient of $t^n/n!$ in $\exp(-t)/(1-t)$, or

$$n!\{1 - 1 + 1/2! - 1/3! + \ldots + (-)^n/n!\};$$

a result already known from another stand-point.*

And similarly, when $x = 2$, the number of positions with no duad is

$$n!\{1 - 1/2 + 1/2!\,2^2 - 1/3!\,2^3 + \ldots\},$$

the series containing $n/2 + 1$ or $(n+1)/2$ terms according as n is even or odd—a result possibly new.

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*Chrystal, Algebra, vol. 2, p. 25; Whitworth, Choice and Chance.

Suppose, to change the mode of illustration, n letters written and their n envelopes addressed, and let the letters be put, at random, one into each envelope. The chance that in no case will each of two people get the other's letter is

$$1 - 1/2 + 1/2!\,2^2 - 1/3!\,2^3 + \ldots,$$

which, when $n = \infty$, is $\exp(-1/2)$.

When $x > n/2$, the number of positions with no x-ad takes the simple form $n!(1 - 1/x)$, so that the number of positions with such an x-ad is $n!/x$, a result which can also be proved directly from (4). For if we denote the left side of (4) by Σ_n, the number sought is

$$n!\,\Sigma_{n-x}/x,$$

and

$$\Sigma_{n-x} = 1.$$

From the scheme (5) it is easy to prove the formula which Rothe (loc. cit.) stated without proof, namely, that if u_n be the number of symmetrical positions,

$$u_n = u_{n-1} + (n-1)u_{n-2}.$$

For since we are to have only monads and duads, u_n is the coefficient of $t^n/n!$ in the power-series for $\exp(t + t^2/2)$, that is,

$$\exp(t + t^2/2) = \Sigma u_n t^n/n!,$$

whence, after differentiation,

$$(1 + t)\Sigma u_n t^n/n! = \Sigma u_n t^{n-1}/(n-1)!,$$

and the required result is obtained by equating the coefficients of $t^{n-1}/(n-1)!$

Again, to determine the number of distinct terms in a symmetrical determinant, we observe that $2^{n_3+n_4+n_5+\cdots}$ positions give terms which are not distinct, inasmuch as we can reflect any cycle with regard to the diagonal without altering the term (Cauchy, Works, series 1, vol. 6, p. 98). That is, when $x > 2$ each x-ad in any given kind of position is to halve the number of positions of that kind. Thus the required number is the coefficient of $t^n/n!$ in

$$\exp\,(t + t^2/2 + t^3/6 + t^4/8 + \ldots),$$

that is, in

$$\exp\,(t/2 + t^2/4)\,.\,\exp\{-\tfrac{1}{2}\log\,(1 - t)\},$$

that is, in

$$\exp\,(t/2 + t^2/4)/\sqrt{1 - t},$$

which is the result obtained by Cayley. See Salmon's Higher Algebra, art. 45 of the third edition.

HAVERFORD COLLEGE, PA.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, February 3, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Professor L. C. Walker, St. Lawrence University; Miss Ruth Gentry, Bryn Mawr College; Miss Frances Hardcastle, University of Chicago. Professor Fine made some remarks upon the continuity of the number system, and Dr. Fiske described several different demonstrations of Weierstrass's theorem that only algebraic and periodic functions can possess an algebraic addition theorem.

THE address of Professor Newcomb on "Modern mathematical thought," which appeared in the BULLETIN for January, has been published in full in *Nature* of February 1, 1894, pp. 325–329.

A PORTION of the mathematical models and charts exhibited at Chicago by the German universities was secured by the department of mathematics of Columbia College. This portion, the principal part of which came originally from the institute of technology at Munich, is illustrative of theory of functions, analysis situs, plane curves and their singularities, surfaces, their singularities and curvature, and line-geometry.

THE last number of the *Rendiconti del Circolo Matematico di Palermo,* vol. 7, No. 6 (November–December, 1893) devotes 58 pages to lists of the mathematical articles contained in the recent volumes of 106 periodical publications. A supplement to the same number contains an index by authors to the first six volumes of the *Rendiconti,* a symbol being affixed to the title of each article in accordance with the notation adopted in the "Répertoire bibliographique des sciences mathématiques."

AT the University of Königsberg Professor H. Minkowski, recently of Bonn, has been made professor of mathematics, and Dr. V. Eberhard has been made professor extraordinarius of the same subject.

AMONG the courses announced at the *Sorbonne* by the Faculté des Sciences for the first semester of the present scholastic year are the following: Appell, Composition of forces and general laws of equilibrium and motion. Boussinesq, Interior friction of fluids; gradual extinction of waves. Darboux, Theory of rectilinear congruences and of the infinitesimal deformation of surfaces. Picard, Theory of differential equations; case of real variables and functions. Poincaré, Analytical theory of the conduction of heat. Tisserand, Theories relative to the figure of heavenly bodies. Supplementary courses and conferences upon mathematical subjects are also given by Andoyer, Kœnigs, Painlevé, Pellat, Puiseux, and Raffy. Among the courses during the same semester at the *Collège de France* are one by Jordan on Elliptic functions and one by M. Lévy on the Theory of the tides.

T. S. F.

AT the mathematical club and seminar of the University of Chicago the following papers have been given since last October: (1) On the history of the addition theorem of elliptic integrals, Professor Bolza. (2) Concerning Del Pezzo's five-cusped plane quintic curve, Professor Moore. (3) A study of certain special cases of the hypergeometric differential equation, Dr. Boyd. (4) The existence of transcendental quantities, Dr. Young. (5) A configuration of 140 lines in space representing the substitution-group of 7 letters, Professor Maschke. (6) Cantor on the existence of transcendental numbers, Professor Bolza. (7) Hölder's proof that the function $\Gamma(x)$ satisfies no algebraic differential equation, Professor Moore. (8) A theorem of Runge in the theory of functions of a complex variable, Mr. Slaught. (9) The canonical coördinates of Hamilton-Jacobi in the differential equations of the problem of three bodies, Dr. Laves. (10) The singularities of surfaces with illustrations from the models of the department, Professor Maschke.

AT the University of Chicago during the summer quarter of 1894 the following courses in higher mathematics will be offered: Professor Moore: (1) Elliptic functions; based on Weber, and presupposing a knowledge of the elements of the general theory of functions of a complex variable. (2) Theory of functions of a complex variable; based on Forsyth. (3) Special seminar on functions; fortnightly; in connection with and a part of courses (1) and (2). Dr. Young: (4) Theory of numbers. (5) Elements of the theory of invariants with applications to higher plane curves.

The departmental club and seminar meets fortnightly. These announcements are preliminary and subject to change to meet more exactly the wishes of prospective students. It is important that those expecting to attend should give early notice. Revised announcements will appear in the May calendar of the university. E. H. M.

A NEW journal is to be published by the Association for the Improvement of Geometrical Teaching. It will appear three times a year, under the name *The Mathematical Gazette.* The editor is Mr. E. M. Langley. The first number will contain, besides questions and solutions, articles on the following subjects: (1) Boscovich's treatment of the conic by means of the eccentric angle; (2) Herbart's view of the place of mathematics in education; (3) Greek geometers antecedent to Euclid; (4) Arithmetical approximation.

EUGÈNE CATALAN, professor of higher analysis in the University of Liége, Belgium, died at Liége on the 14th of February. A. Z.

IN the preface to the recently published third volume of Lie's "Theory of transformation groups," edited by Engel, is a notice of two new works which Lie proposes to undertake. One is to give a detailed account of all his geometrical investigations and will relate to group theory, contact transformations, and differential equations. This is to be edited by Dr. Scheffers.

The other is to be edited by Professor Engel. It will present an independent theory of differential invariants, and, taking this as a basis, will attempt to develop in a perfectly complete and general manner the theory of continuous groups defined by differential equations. In this treatment infinite and finite groups will be equally considered. In the theory of differential equations the following problems will be discussed: first, the establishment of integrable canonical differential equations; second, the determination of criteria by which the possibility of reducing a given differential equation to a canonical form may be proved; third, a method of performing this reduction. A thorough investigation will be given also of those problems in integration which can be reduced to the integration of a complete system admitting known infinitesimal transformations. E. M. B.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bianchi (L.). Lezioni di geometria differenziale. Parte I., pp. 1-256. Pisa, 1893. 8vo. Complete in two parts. Fr. 21.00

Cajori (F.). A history of mathematics. New York, Macmillan, 1894. 8vo. 14 and 422 pp. Cloth. $3.50

Cesàro (E.). Corso di analisi algebrica con introduzione al calcolo infinitesimale. Torino, Bocca, 1894. 8vo. 8 and 499 pp. Illustrated. Fr. 12.00

Dirichlet, see **Lejeune-Dirichlet** (P. G.).

Fouret (G.). Notions géométriques sur les complexes et les congruences de droites. Paris, Gauthier-Villars, 1893. 8vo. 74 pp.

Jacob (J.). Zur Lehre von der Theilbarkeit der Zahlen. [Progr.] Mährisch-Neustadt, 1893. 8vo. 7 pp.

Killing (W.). Einführung in die Grundlagen der Geometrie. Vol. I. Paderborn, Schöningh, 1894. 8vo. 10 and 357 pp. Illustrated. Mk. 7.00

Lejeune-Dirichlet (P. G.). Vorlesungen über Zahlentheorie. Herausgegeben und mit Zusätzen versehen von R. Dedekind. 4te, umgearbeitete und vermehrte Auflage. Braunschweig, Vieweg, 1894. 8vo. 17 and 657 pp. Mk. 14.00

Lindenthal (E.). Die Sprache der Mathematik. [Progr.] Wien, 1893. 8vo. 19 pp.

Loria (G.). Della varia fortuna di Euclide in relazione con i problemi dell' insegnamento geometrico elementare. Roma, Tip. Elzeviriana, 1893. 8vo. 37 pp.

Lucas (Ed.). Récréations mathématiques. Vol. III. Paris, Gauthier-Villars, 1893. 8vo. 200 pp. Fr. 6.50

Macfarlane (A.). On the definitions of the trigonometric functions. Boston, Cushing [1894]. 8vo. 49 pp.

Méray (C.). Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques. (In 3 vols.) Vol. I.: Principes généraux. Paris, Gauthier-Villars, 1894. 8vo. 23 and 405 pp. Fr. 13.00

Périer (G.). Sur une équation différentielle du troisième ordre. [Thèse.] Paris, Nony, 1893. 4to. 135 pp.

Study (E.). Sphärische Trigonometrie, orthogonale Substitutionen und elliptische Functionen. Leipzig, Hirzel, 1893. Mk. 5.00

Wiskoczil (E.). Direkte Construction der sechs Kegelschnitte, welche mit zwei gegebenen Kegelschnitten in doppelter Berührung stehen und welche überdies noch eine gegebene Gerade berühren, vorausgesetzt, dass die Gerade die zwei Curven schneidet. [Progr.] Iglau, 1893. 8vo. 13 pp. 1 plate.

II. ELEMENTARY MATHEMATICS.

Bachelet (A.). Nozioni di geometria elementare. 4ta ed. riveduta, con modificazioni e aggiunte in relazione ai programmi liceali ed una serie ordinata di esercizi. Torino, Paravia, 1894. 16mo. 302 pp. Illustrated. Fr. 3.50

Burat (E.) Cours d'arithmétique élémentaire, à l'usage des écoles primaires et des classes de grammaire des lycées et collèges, rédigé d'après le programme d'enseignement pour les écoles publiques du département de la Seine, ouvrage renfermant de nombreux exercices et les premières notions de géométrie. 19th ed. Paris, Belin, 1893. 12mo. 390 pp. Illustrated.

Dobbie (A. B.). A text-book of solid or descriptive geometry, including elementary courses on plane geometry and graphic arithmetic. London, Blackie, 1894. (Science text-books.) 8vo. 230 pp. 2s. 6d.

Laurent (H.). Traité d'algèbre, à l'usage des candidats aux écoles du gouvernement. 5me édition, en harmonie avec les nouveaux programmes, revue par J. H. Marchand. Troisième partie, à l'usage des classes de mathématiques spéciales: théorie des equations. Paris, Gauthier-Villars, 1893. 8vo. 215 pp. Fr. 4.00

Love (J. L.). Outlines of plane trigonometry. Cambridge, Harvard University Press, 1894. 16mo. 24 pp. $0.20

Rosenberg (K.). Die Schnittcurven einer Ebene mit einem Rotationskegel. [Progr.] Wien, 1893. 8vo. 5 pp. Illustrated.

Schotten (H.). Inhalt und Methode des planimetrischen Unterrichts. Eine vergleichende Planimetrie. Zweiter Band. Leipzig, Teubner, 1894. 8vo. 4 and 410 pp. Mk. 8.00

Smith (J. Hamblin). Elementary algebra. New and revised ed. London, Longmans, 1894. 12mo. 400 pp. With answers. 3s. 6d.

Sterza (A.). Aritmetica pratica per il ginnasio inferiore. Mantova, Mondovi, 1893. 8vo. 146 pp. Fr. 1.50

——— Aritmetica pratica e nozioni elementari di geometria per le tre classi del corso preparatorio alle scuole normali. Mantova, Mondovi, 1893. 16mo. 210 pp. 8 plates. Fr. 2.00

——— Aritmetica razionale per il ginnasio superiore. Mantova, Mondovi, 1893. 16mo. 172 pp. Fr. 1.50

III. APPLIED MATHEMATICS.

Fabry (L.). Étude sur la probabilité des comètes hyperboliques et l'origine des comètes. [Thèse.] Marseille, Barlatier et Barthelet, 1893. 4to. 219 pp.

Gelcich (E.). Cartografia. Manuale teorico-pratico, con un sunto sulla storia della cartografia. Milano, Hoepli, 1894. 24mo. 7 and 257 pp. Illustrated. (Manuali Hoepli.)

Heaviside (O.). Electro-magnetic theory. Vol. I. London, The Electrician, 1894. 8vo. 480 pp. 12s. 6d.

Hertz (H.). Electric waves: being researches on the propagation of electric action with finite velocity through space. Authorized English translation by D. E. Jones. With a preface by Lord Kelvin. London, Macmillan, 1894. 8vo. 8 and 278 pp. Cloth. $2.50

Längengradmessung, Die Europäische, in 25 Grad Breite von Greenwich bis Warschau. (Veröffentlichung des K. Preuss. Geodätischen Institutes und Centralbureaus der internationalen Erdmessung.) Herausgegeben von F. R. Helmert. Heft I: Hauptdreiecke und Grundlinienanschlüsse von England bis Polen. Berlin, 1893. 4to. 7 and 263 pp. 2 plates. Mk. 18.00

Lesky (A.). Die historische Entwicklung des Problems der Saitenschwingung. I. [Progr.] Graz, 1893. 8vo. 27 pp.

Lockyer (J. N.). The dawn of astronomy: a study of the temple worship and mythology of the ancient Egyptians. London and New York, Macmillan, 1894. 8vo. 432 pp. Illustrated. Cloth. $5.00

Massau (J.). Traité de mécanique. Gand, Lobel, 1894.

Newton (H. A.). On the capture of comets by planets, especially their capture by Jupiter. (National Academy of Sciences, Vol. 6, 1893.) 4to. 28 pp.

Poynting (J. H.). The mean density of the earth: an essay to which the Adams prize was adjudged in 1893 in the University of Cambridge. With illustrations and 6 folding plates. London, Griffin, 1894. 8vo. 176 pp. 12s. 6d.

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THE THEORY OF FUNCTIONS.

A Treatise on the Theory of Functions. By JAMES HARKNESS and FRANK MORLEY. New York, Macmillan & Co., 1893. 8vo. IX + 503 pp.

IN the vast realm of Mathematical Science, as it stands before us to-day, there exists, owing to the genius, activity, and assiduity of mathematicians, such a great variety of widely extended subdivisions that it seems to be an impossibility for one man to keep pace with the rapid development of all of them. The consequence has been that mathematicians try to confine themselves to some special line of their science, not only in research work, but also in reading up mathematical literature. This is a pity, because the cultivation of any special branch suffers considerably by disregarding the methods and results of other branches, while, on the other hand, the most beautiful results have often been obtained by the combination of apparently unconnected regions: it is to be regretted, but it is a fact, and will remain so as long as the capacity of the human intellect remains limited.

There exists, however, one branch of mathematical science whose bearing on nearly all the other parts is so evident that it appears to be, with perhaps a few exceptions, indispensable for special work of any kind—that is, the *Theory of Functions*. Even in applied mathematics it seems to gain a foothold: the theoretical astronomer as well as the mathematical physicist has frequently enough, when he wants to feel safe ground under his feet in his calculations, to fall back on the theory of functions.

While thus the high importance of function theory will be readily admitted, it has until lately been impossible to obtain a fair knowledge of this subject without consulting quite a number of different German or French treatises and original papers. There did not exist any English text-book on the subject at all.* But there is also no Continental work which deals with the theory of functions in any degree of completeness. Either Cauchy's, or Riemann's, or Weierstrass's method is given alone, whereas a combination of all three methods is needed in order to pursue the study of the subject to advantage.

An urgent demand was met therefore when there appeared, some months ago, two new, comprehensive treatises, both in English: Forsyth, "Theory of Functions of a Complex Variable;" and the work of which we shall try to give a short review in the following pages: Harkness and Morley, "A Treatise on the Theory of Functions."

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* I should perhaps except here some chapters in Chrystal's Algebra, which bear on subjects belonging to the theory of functions.

The first thing which strikes the reader is the richness of material contained in the work. Besides an exhaustive presentation of the principal conceptions and deductions as they are given by the great founders of the theory, Cauchy, Riemann, and Weierstrass, we find in the book a full account of Weierstrass's theory of elliptic functions, of double Theta-functions, and of Abelian integrals and functions, including the difficult and intricate proofs of existence theorems, leading up to the most recent investigations, and foreshadowing further developments in this direction. Also some collateral branches of the theory are taken up, as, for instance, algebraic curves and (non-analytic) functions of a real variable.

And this rich material is not merely compiled in a loose manner, but it has been worked over carefully, and systematized in a way which gives the book a thoroughly scientific character. Even in those parts which treat of the most difficult problems of the theory the impression is forced upon the reader that he is guided by the hand of men who are thoroughly versed in modern mathematical thought.

An added value is conferred upon the work by the fulness of references which are given, not only throughout the whole volume, but also at the end of each chapter in a separate appendix. This should be done in every scientific treatise, for, besides its immediate purpose, it always exerts an inspiring influence upon the student, and it leads him to a principle only too often violated: "*Never study one book only.*"

As to the manner in which the subject is presented to the reader, it seems to be natural that in a work like this the language cannot be otherwise than concise, sometimes summary. The conciseness is, however, in some places carried so far as to mislead a beginner. To make up for compactness of expression, all should have been done to facilitate a clear understanding of the text. Unfortunately this is not always the case. The theorems enunciated are not sufficiently, or even not at all, separated from the context. Thus it frequently happens that the reader cannot decide where the enunciation of a theorem ends and the explanation of it begins. Definitions of vital importance are given in the middle of a paragraph without being made prominent in any way. Much of this could have been avoided very easily by the use of italics.

Furthermore it would have added considerably to perspicuity if, after long deductions, the result of the investigation were expressly stated. As it is, the reader has in many paragraphs to find out for himself what the authors have in mind. This may perhaps be less felt by a reader who studies the volume word by word, but it will prove a more serious drawback for one who wants to use the book for reference, on account of the difficulty in ascertaining at a rapid glance the bearing of the different paragraphs on the subject in question.

It is to be hoped, however, that these defects will be removed in a second edition.

Chapter I., "*Geometric Introduction,*" opens with the ordinary geometrical representation of a complex variable in the plane. In order to guard against useless generalizations, it is shown in § 15 that an extension to three dimensions becomes illusory if the laws of ordinary algebra shall be maintained.

A function w of a complex variable z being defined at first merely as a correspondence, the definition of continuity of such a function is given in § 16, and illustrated by simple examples.

In the sequel it becomes necessary to speak also of branches of a many-valued function; § 18 deals therefore with Cauchy's proof of the fundamental theorem of algebra. Now this proof, as is known, shows only that the modulus of an integral function possesses the lower limit—zero. This point should be emphasized *somewhere* in the book, if not in the introductory chapter, certainly later on, after it has been shown that a continuous function always attains its limit.

The important criterion of a monogenic function w of z, i.e., a function whose derivative $\frac{dw}{dz}$ is independent of dz, is developed in § 20, and in connection therewith numerous and very instructive examples are given,—in §§ 21–30 for one-valued, in §§ 46–48 for many-valued, functions. §§ 29 and 30, which treat of conform representation by fractional linear functions, may serve as well as a preliminary exercise for the study of modular functions.

In the remaining paragraphs the authors give a very elegant exposition of several points in the theory of invariants for cubic and biquadratic binary quantics. For the benefit of those students, however, who are not yet acquainted with invariants, it might be said that these paragraphs are not absolutely required for the study of the following chapters. Perhaps it would have been a better plan to confine the subject to only the most important points, as, for instance, the deduction of g_2 and g_3, and to set forth this deduction in such a way that there would not be presupposed, on the part of the reader, familiarity with the theory of invariants.

Chapter II., "*Real functions of a real variable,*" gives a succinct account of some of the most important recent researches concerning the foundations of the infinitesimal calculus.

Since the time when Abel wrote, in his well-known letter to Hansteen, "Je consacrerai toutes mes forces à répandre de la lumière sur l'immense obscurité qui règne aujourd'hui dans l'analyse," much has been done in the direction indicated by the illustrious mathematician. The "immense obscurity" has been traced back to its source, viz., a comparatively small number of fundamental conceptions which seem simple enough as long as we are satisfied with geometrical evidence, but which turn out to be full of the most intricate difficulties as soon as they are exposed to the search-light of purely analytical methods. Such are the conceptions of: *Irrational numbers* (which may be said to be at the bottom of all difficulties of the infinitesimal calculus); that of a *limit* with its delicate distinctions between upper limit and maximum, between uniform and non-uniform convergence; the general idea of a *function,* with its wide range of possibilities. These and other difficulties have been cleared up by the combined efforts of men like *Abel, Cauchy, Bolzano, Weierstrass, du Bois-Reymond, Cantor,* and others, and the infinitesimal calculus has been put upon a basis as sound and solid as the foundation of arithmetic.

The importance of these researches to the theory of functions of a complex variable can hardly be overestimated; for the same difficulties that face us in the elements of real infinitesimal calculus will turn up over and over again, whether we consider Cauchy-Riemann's or Weierstrass's theory. As examples, I may mention Goursat's proof of Cauchy's theorem and Weierstrass's theorem on the existence of singular points on the circumference of the circle of convergence.

There is still another, more pedagogical, reason why a discussion of functions of a real variable should not be omitted in a treatise on analytic functions. We understand much better the properties of these latter functions and the consequences of the restrictions imposed upon them by their very definition when we have already obtained an insight into the many difficulties by which non-analytical functions are surrounded.

We are therefore indebted to the authors for having given to this important subject a place in their volume.

We find in §§ 50–54 an exceedingly lucid account of Cantor's definition of irrational numbers, which is followed by definitions of upper and lower limits, points of accumulation, mass of points, and Cantor's derived mass of points.

All this may be called rather a theory of the *independent* variable; but certainly this theory should always precede a theory of the *dependent* variable, i.e., a theory of functions.

In §§ 60–64 important definitions and theorems on upper and lower limits of functions and on continuity of functions are deduced. Weierstrass's famous example illustrates in § 65 the possibility of a continuous function which has nowhere a differential quotient.

Chapter III. "The theory of infinite series."—This chapter, forming the foundation of function-theory, may be divided into two parts: (1) Infinite series and products; (2) Weierstrass's theory of analytic functions.

It was felt in many researches that the old definition of a function as a dependence of one variable on the other was too vague to make functions accessible to analytic investigation. In order to do the latter, it proved to be necessary to impose some restrictions upon this very general idea of a function, to limit its wide range in such a way as to attain, if possible, concrete mathematical expressions. This has been done, as is well known, by Cauchy, who, *confining* himself to mongenic functions, arrived at the expansion of such a function into a power-series.

Weierstrass's method is just the opposite. He starts from given functions, well defined by simple analytic expressions, and, never deviating for a moment from the safe ground of a concrete mathematical representation, *amplifies* and generalizes these functions in such a way as to cover as much as possible the ample ground occupied by the general notion of a function. Thus he reaches, from precisely the opposite end, not only connection with Cauchy's theory, but he also comes, by his analytic functions, as near as one pleases—in a strictly mathematical sense—to every continuous non-analytical function.*

Turning now to the first part of this chapter, we find in §§ 67–69 a statement of the principal definitions and theorems on series with real terms. As to the proofs, reference is made to the usual text-books.

Series with complex terms are considered in §§ 70–73, including in particular the definition of uniform convergence. Examples are given also for *non*-uniform convergence. This is the more instructive case, since the student will be inclined to think, at first, that convergence always implies uniform convergence.

In the following sections on multiple series there is of special interest the example taken from the theory of elliptic functions:

$$\Sigma\Sigma'\frac{1}{(m_1\omega_1 + m_2\omega_2)^\lambda}.$$

Two proofs are given for the convergence of this series for $\lambda > 2$, the second being an extension to double series of "Cauchy's integral test for the convergence of simple series."

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* See page 48 of the "Evanston Colloquium," by Felix Klein. New York, 1894.

It would be very desirable if this test for simple series* were given *somewhere* in the book—which is not the case,—the more so as the account of its extension to double series is extremely succinct. The same extension recurs in chapter VIII., § 223, where the condition for the convergence of double Theta-series is deduced.

The second part of this chapter, §§ 82–107, affords a comprehensive and lucid introduction into Weierstrass's theory of analytic functions. Instructive illustrations are given of the continuation of a power-series, the influence of singular points on this process, and especially of the case where the region of the analytic function is limited by a closed line (§§ 103 and 104).

Chapter IV., "*Algebraic functions*," deals with the expansions of algebraic functions in the vicinity of branch-points and singular points, and may be regarded as an essential preparation for the following chapters on Riemann's theory. The discussion here presented to the reader, which is generally omitted in other treatises on function-theory, confers a special value on this chapter.

It is important for the student to learn the different points of view from which an equation $f(x, y) = 0$ is to be looked at in geometry and in the theory of function. If this equation is of the m^{th} degree in x and of the n^{th} degree in y, it will represent geometrically, in general, a curve of the $m + n^{\text{th}}$ order, so that the line $x = 0$ for instance will have $m + n$ points of intersection with the curve; whereas in the light of function-theory only n values of y correspond to any given value of x.

With regard to singularities, there is also a difference between the views taken in geometry and in function-theory. While the branch-points of the function y, defined as above by the equation $f(x, y) = 0$ as a function of x, turn out to be of paramount importance in the theory of functions,† they are, geometrically, nothing else than the points of contact of tangents parallel to the Y-axis, and therefore of no higher interest than any other system of points of intersection between the curve and the first polar with regard to any point in the plane.

Expansions in the vicinity of singular points, such as nodes, cusps, etc., bear, of course, immediately on geometry, since they determine the geometrical character of the singularity. Interesting examples of this kind are given in § 122.

As to the exposition of the subject, the authors follow, at first, Noether's method of resolving by Cremona-transformations — speaking geometrically — higher singularities with partly coinciding tangents into those whose tangents are all distinct. Also Puiseux's method, emanating from Newton, is described and illustrated by some examples.

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* Cf. Picard's Traité d'Analyse, vol. I., chapter 1, § 17.

† This is not so much the case in Weierstrass's theory.

The remainder of the chapter is devoted to Clebsch-Lüroth's interesting theory of loops. This method of investigation, frequently used by French mathematicians, has, at least for some lines of work, decided advantages, the independent variable moving only in a singly-covered plane. Besides, it can be used for simplifying the study of Riemann's surfaces, as pointed out in chapter VI.

Chapter V. "Integration."—Two ways lie open before the student for entering into the theory of functions. One of them has been described in chapter III. The other is afforded by Cauchy's beautiful method of integration along a closed line in the plane of complex numbers.

It cannot be denied that this second way is a shorter one, but only as long as certain fundamental points, which seem at a first glance to be evident, are taken for granted. As soon, however, as absolute rigor is required, the number of those fundamental difficulties bearing on the foundation of the infinitesimal calculus, which have been pointed out in the review of chapter II., will prove to be much greater in Cauchy's than in Weierstrass's theory. On the other hand, Cauchy's methods, important and attractive in themselves, are indispensable in Riemann's theory of Abelian integrals.

In the present chapter Cauchy's method is developed and carried on until connection with Weierstrass's theory is reached.

Among the different proofs of Cauchy's fundamental theorem Goursat's method has been chosen as the one which makes immediate use of the existence of a differential quotient and which does not apply double integration. The presentation of this proof being exceedingly concise, the student should consult, besides, the original paper of Goursat* and one of the other proofs referred to. Riemann's classical proof is given in chapter VI., and may also be consulted, as it takes only a singly-sheeted plane instead of the general Riemann surface.

In §§ 135–141 the important applications of Cauchy's theorem to integration about singular points are discussed, leading to Taylor's and Laurent's theorems.

The connection with Weierstrass's theory being thus established, we find in the following paragraphs those fundamental theorems which refer mostly to analytic functions whose singularities are known,—in particular a comprehensive account of

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* Cf. also *Franklin:* "Two proofs of Cauchy's theorem." *American Journal of Math.*, vol. IX., p. 389.

Mittag-Leffler's theorem, and Weierstrass's factor-representation of a function with assigned zeros.

Applications to the calculation of definite integrals and a succinct account of Cauchy's proof of the existence of integrals of differential equations close the chapter.

Chapter VI., "*Riemann surfaces.*"—The first half of this chapter, being devoted to the theory of Riemann surfaces proper, §§ 158–173, can be highly recommended as a lucid and pedagogical introduction into the subject.

At first the principal ideas of Riemann surfaces are developed, leaving aside all generalities. A number (9) of very instructive, well-chosen, and elaborated examples showing how to construct a Riemann surface will prove to be of extremely good service to the student.

Theorems on the connection of surfaces in general and of Riemann surfaces in particular are given in §§ 168–170, leading to the definition of the deficiency p.

It must be put down as, in a certain sense, a drawback of Riemann's theory that these geometrical considerations on the connection of surfaces are, in the general case, and if any degree of reliability in the conclusions is to be reached, of a highly complicated character, and lead into abstruse difficulties. Now, certainly, no mathematician should object to mathematical difficulties; but these difficulties, as they are presented here, bearing mostly on "analysis situs," lie in a region which is entirely incongruous with pure analysis.

If, however, the geometrical theorems in question have been proved or are taken for granted, the beauty of Riemann's theory lies open before us. No simpler and clearer insight into the real meaning and nature of the deficiency p, for instance, can be gained than the perception that p constitutes the one and only invariant of the surface with regard to continuous deformation.

The canonical dissection of a Riemann surface is based in § 171 upon the theory of loops discussed in chapter IV. Also, Klein's normal surface and its dissection, which exhibits so very clearly the true nature of the problem, is presented to the reader, as well as Riemann's original method.

The subject of the second part of this chapter is, at first, the study of functions on a Riemann surface. Integrals on the surface are then discussed at some length, but only as far as to the demonstration of periods, a further development being the chief object of chapter X.

There follows now in §§ 183–190 an investigation which is of a special theoretical value. The theory of algebraic functions, as it has been developed by Kronecker and Dedekind-Weber, is sketched briefly in order to show that a curve $f(w, z) = 0$ can be transformed rationally in such a way that the final equation does not contain any higher singularities than simple nodes. The presentation of this subject is very succinct, and can hardly be regarded as more than a guide through Kronecker's and Dedekind-Weber's papers.

The chapter concludes with the explanation of Klein's interesting transformation of a curve of class 3, $p = 1$, into the normal surface of deficiency 1, that is, into an anchor-ring.

Chapter VII. "*Elliptic functions.*"—The subject presented in this chapter is, in essence, Weierstrass's theory of elliptic functions.

The definition of primitive periods is given in §§ 191–194 in a very clear and exhaustive manner.* Next the function $\wp(u)$ is introduced immediately, defined by the series

$$\frac{1}{u^2} + \Sigma'\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right),$$

and the double periodicity of this function demonstrated. Liouville's theorems on doubly periodic functions follow in §§ 196–200. In § 201 the $\wp$-function is taken up again, the σ-function deduced from it and then the theory is carried through, essentially according to Halphen and Schwarz's Formelsammlung.

We regret that the authors did not follow Weierstrass's construction of the elliptic functions as the most general one-valued functions which possess an algebraic addition-theorem, although they refer to it in several places. A few steps would have been sufficient, since nearly all the auxiliary theorems are at hand (see chapter V., § 151), in order to deduce this beautiful theory, one of the masterpieces of Weierstrass's creations. No better illustration of the high practical usefulness of those general theorems on the theory of functions developed in chapter V., which may seem to the student somewhat abstract in their generality, could have been given.

Some remarks may be made with regard to notation. Weierstrass denotes his fundamental periods by 2ω, $2\omega'$, using afterwards, where the sum $\omega + \omega'$ comes in, the following notations:

$$\omega_1 = \omega, \quad \omega_2 = \omega + \omega', \quad \omega_3 = \omega'.$$

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* It may only be mentioned that it would perhaps have been better to define the parallelogram of periods *ab initio* as the totality of all the points within and on the rim of the parallelogram *OABC*, *except the points on AB and BC.*

Instead of this, Harkness and Morley introduce this notation:

$$\omega_1 = \omega,\ \omega_2 = -\omega - \omega',\ \omega_3 = \omega'.$$

Hence the relation holds:

$$\omega_1 + \omega_2 + \omega_3 = 0.$$

The symmetry of the formulæ obtained thereby will sufficiently justify this deviation from the original notation.*

Chapter VIII. "Double Theta-functions."—One who wishes to make his first acquaintance with double θ-functions cannot do better than to study first this chapter before taking up some of the more detailed treatises or original papers on the subject. For the benefit of some readers it might be added that there is not presupposed any knowledge of the preceding parts of the volume.

Besides the definitions and fundamental properties of double θ-functions, and a very valuable digest and explanation of the different notations, by marks, current indices, and duads, the theory is developed so far as to embody the most remarkable linear, quadratic, and bi-quadratic relations between θ-functions.

Some original work has been added to the subject by the authors. In the so-called θ-formula of Riemann, extended by Prym-Krazer, there occurs a constant factor whose numerical value ($= 4$) is generally computed by expansion into series.† The authors have determined this factor by a different elegant method, which proves to be effective also for the deduction of quite a number of θ-relations.

In the concluding paragraph a brief definition of p-tuple θ-functions is added. It seems somewhat strange that the analogous exponential representation of ordinary elliptic θ-functions is nowhere to be found in the volume. These functions are mentioned only briefly in chapter VII., where they are derived in trigonometrical form from the σ-functions.

Chapter IX. "Dirichlet's problem."—The most characteristic feature of Riemann's magnificent theory of Abelian functions is that he starts, not from an algebraic equation, but from a Riemann surface. On this surface—made simply connected by a system of cross-cuts—he defines and proves the existence of "functions w" of $z = x + iy$, having given algebraic or logarithmic discontinuities and constant moduli of periodicity. With the help of these functions w he is able to build up "functions s" which are single-valued in the original surface T, and have no other than algebraic discontinuities. The functions s are proved to be algebraic functions of z, while the functions w turn out to be integrals of rational functions of s and z, so-called Abelian integrals.

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* *Study* in his "Sphaerische Trigonometrie" (Leipzig, 1893), has been led to the same change of notation by similar reasons (see l. c., page 189).

† Cf. *Krause:* Die Transformation der hyperelliptischen Functionen, p. 46.

The theorem of the existence of the functions w is therefore of paramount importance for Riemann's theory. Riemann's own proof is based on Dirichlet's principle, viz., the conclusion that among all functions u of x and y satisfying certain boundary- and continuity-conditions, there exists one for which the integral

$$\int_{(T)}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]dT$$

takes a minimum value. This being once granted, it follows at once from the elements of the calculus of variations that the minimizing function satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Dirichlet's conclusion lies, however, open to serious objections, since the possibility is not taken into account that the integral might be made to tend towards zero by functions u which approach functions no longer satisfying the conditions imposed upon u. The conclusion is, in fact, one of the many examples where the distinction between lower limit and minimum is neglected, and it is entirely analogous to the conclusion, known to be wrong, that, the sum of the angles of a spherical triangle always being greater than π, there must exist a triangle for which the sum of the angles is a minimum.*

This gap in Riemann's theory has been filled by Neumann, Schwarz, and others, who have given rigorous proofs of Riemann's existence theorems independent of Dirichlet's conclusion. These proofs form the subject of chapter IX., whose length is entirely in proportion with the importance of the subject, though some readers may perhaps wish it were shorter.

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* Weierstrass gave this example in one of his lectures.

The problem in its simplest form is stated as follows: "To find a function $u(x, y)$ which, together with its differential quotients of the first two orders, shall be one-valued and continuous in a region T, which shall satisfy Laplace's equation and shall take assigned values upon the boundary of the region."

After the demonstration of two theorems of Painlevé's on rim-values and Green's theorem for two variables, with those consequences which bear on the subject, Schwarz's solution of the problem is deduced for the simple case where the given region T consists of a circle and the function on its circumference is supposed to be continuous.

It is shown next—leaving aside some allied investigations—how these restrictions can be removed. Singularities of a certain kind are admitted on the circumference, and—by the help of conform representation and Schwarz's alternating process—the solution is extended to regions bounded by any analytic lines.

The following sections are devoted to Harnack's investigations, which lead, on the basis of Schwarz's important results and by the help of Green's function g to the solution of the problem for a simply connected region whose rim consists of an integrable curve, and has only a finite number of changes of direction.

The last section of the chapter extends the solution to the case where interior discontinuities are given of the kind met with in the discussion of Abelian integrals.

The reader who wishes to become familiar with the subject will have some difficulty in working his way through this chapter, but the subject offers so many intricate difficulties in itself, that he will find it necessary to fall back on the study of at least some of the original papers.

From the solution of Dirichlet's problem Riemann's existence-theorems can be derived without further difficulties. They are given at the end of the following chapter, in §§ 306 and 307. Klein's methods of proof, as exhibited in Klein-Fricke's Modulfunctionen, are adopted.

Chapter X., "*Abelian integrals*," gives a detailed exposition of Riemann's theory, with a supplementary sketch of Clebsch-Gordan's work, so that this chapter may be regarded as a continuation of chapter VI. as well as of chapter IV.

Following strictly Riemann's ideas, the surface T is made the starting-point, and accordingly the theory of the integrals precedes that of rational functions on the surface. The properties of the integrals of the first kind, their periods and their derivatives "the functions Φ," are studied in detail in §§ 277–281.

Next follow the integrals of the second kind, with their important application to the construction of rational functions with arbitrarily assigned zeros and poles, leading to Riemann-Roch's theorem.

In this connection a short account of Riemann's theorems on the class-moduli is added. The elementary integrals of the third kind, their periods, and the theorem on the interchange of limits and parameters are treated in § 285, while § 286 deals with the most general Abelian integral.

After a digression on Clebsch and Gordan's method of homogeneous variables in §§ 287–290, Abel's theorem is taken up in § 291, and two proofs are given—Neumann's and Riemann's. Then follow Riemann's theorems on zeros of the function $\theta(u_1 - e_1, \ldots u_p - e_p)$. The determination of the constants k is discussed at length, and Riemann's as well as Clebsch and Gordan's choice of the lower limits, by means of the contact-curves of order $n - 2$, is explained (§§ 294–300).

A short sketch of Jacobi's inversion-problem and Clebsch-Gordan's solution of it close the general part of the chapter, while the remaining paragraphs contain a very interesting and instructive application of the preceding general theories to the case $p = 2$. As normal curve a nodal quartic is taken, which lends itself more readily than the hyperelliptic curve to Clebsch's method. The determination of the constants k leads to the important correspondence between the "transcendental" and "algebraic" characteristic of a theta-function which throws a new light upon the duad-notation explained in chapter VIII.

Finally, Klein's expression of the even and odd double θ-functions in terms of algebraic functions and of a single integral of the third kind is derived, and the reader thus finds himself at the end of the chapter landed in the midst of the most important recent discoveries on Abelian functions.

The whole chapter is not only of high scientific value, but it is also well arranged and very clearly written, and furnishes an excellent introduction into the theory of Abelian functions.

Harkness and Morley's treatise has rendered the theory of functions accessible to every one who wishes to acquire a thorough knowledge of the subject. The great merits of this valuable work will secure it a high rank in modern mathematical literature.

H. MASCHKE.

UNIVERSITY OF CHICAGO, *March*, 1894.

NOTE ON SUBSTITUTION GROUPS OF EIGHT LETTERS.

BY G. A. MILLER, PH.D.

I HAVE found a non-primitive group of degree eight and order twenty-four which is not contained in Professor Cayley's list, vol. 25, p. 140, *Quarterly Journal of Mathematics*, nor in Professor Cole's supplementary list published in the BULLETIN, vol. 2, pp. 187 and 189. The group may be written in the form

$$(abcd\,.\,efgh)\ \text{pos.}\ (ae\,.\,bg\,.\,cf\,.\,dh).$$

The substitutions are

$$\begin{array}{llll}
1 & abc\,.\,efg & ab\,.\,cd\,.\,ef\,.\,gh & afch\,.\,bgde \\
 & abd\,.\,efh & ac\,.\,bd\,.\,eg\,.\,fh & ahcf\,.\,bedg \\
 & acd\,.\,egh & ad\,.\,bc\,.\,eh\,.\,fg & afdg\,.\,bhce \\
 & bcd\,.\,fgh & af\,.\,be\,.\,cg\,.\,dh & agdf\,.\,bech \\
 & bdc\,.\,fhg & ag\,.\,bf\,.\,ce\,.\,dh & agbh\,.\,cfde \\
 & acb\,.\,egf & ah\,.\,bf\,.\,cg\,.\,de & ahbg\,.\,cedf \\
 & adb\,.\,ehf & ae\,.\,bg\,.\,cf\,.\,dh & \\
 & adc\,.\,ehg & ae\,.\,bh\,.\,cg\,.\,df & \\
 & & ae\,.\,bf\,.\,ch\,.\,dg &
\end{array}$$

Two of the six groups of order 16 given by Professor Cole in the BULLETIN, vol. 2, p. 187, are identical, viz.:

$$\begin{array}{ll|ll}
1 & ac\,.\,bd & 1 & eg\,.\,fh \\
ac & bd & ef\,.\,gh & eh\,.\,fg \\
ab\,.\,cd & ad\,.\,bc & efgh & ehgf \\
abcd & adcb & eg & fh
\end{array}$$

and

$$\begin{array}{ll|ll}
1 & ac\,.\,bd & 1 & eg\,.\,fh \\
ac & bd & efgh & ehgf \\
ab\,.\,cd & ad\,.\,bc & eg & fh \\
abcd & adcb & gh & eh\,.\,fg
\end{array}$$

The substitutions of the group of order 12, given just below these six, should be

1
abc . efg
acb . egf
abd . efh
adb . ehf
acd . egh
adc . ehg
bcd . fgh
bdc . fhg
ab . cd . ef . gh
ac . bd . eg . fh
ad . bc . eh . fg

Since two of the published groups of degree eight are identical and a new one is added, their total number is still 199.

It remains to note a typographical error in Professor Cole's supplementary list. The combining substitution in the third group of order 32, p. 188, should be *aecg . bf . dh* instead of *aecg . bfdh*; the latter substitution gives the same group as *ae . bf . cg . dh,* which is already in the list. There is a considerable number of other typographical errors in each of the lists referred to, but they are of such a nature as to be easily detected by those employing the groups and require therefore no special mention.

ANN ARBOR, *March* 12, 1894.

With the above corrections, for which I am under great obligations to Dr. Miller, the question of the possible substitution groups of eight letters is now settled beyond doubt.

F. N. COLE.

ON THE GENERAL TERM IN THE REVERSION OF SERIES.

BY PROF. JAMES McMAHON.

IN reverting the series

$$y = a_0 x + a_1 x^2 + a_2 x^3 + \dots \qquad [a_0 \neq 0]$$

it is usual to assume a development for x in the form

$$x = A_0 y + A_1 y^2 + A_2 y^3 + \dots,$$

and then to substitute, and equate coefficients of like powers, thus determining $A_0, A_1, \dots$ in succession.

This method does not give any observable law for the independent formation of the expression for the coefficient of a given power of y.

A different method, however, based on Lagrange's series, furnishes the desired general term.

The first equation may be written

$$a_0 x = y - a_1 x^2 - a_2 x^3 - \dots,$$

or

$$x = z + b_1 x^2 + b_2 x^3 + \dots, \quad = z + \phi(x),$$

where

$$z = \frac{y}{a_0}, \quad b_1 = -\frac{a_1}{a_0}, \quad b_2 = -\frac{a_2}{a_0}, \quad \dots,$$

and

$$\phi(x) = b_1 x^2 + b_2 x^3 + \dots;$$

whence, by Lagrange's series,

$$x = z + \phi(z) + \frac{1}{2!}\frac{d}{dz}[\phi(z)]^2 + \frac{1}{3!}\frac{d^2}{dz^2}[\phi(z)]^3 + \dots$$

in which

$$\phi(z) = b_1z^2 + b_2z^3 + \ldots,$$

$$\frac{d}{dz}[\phi(z)]^2 = 4b_1^2z^3 + 5(2b_1b_2)z^4 + 6(b_2^2 + 2b_1b_3)z^5 + \ldots,$$

$$\frac{d^2}{dz^2}[\phi(z)]^3 = 6\,.\,5b_1^3z^4 + 7\,.\,6(3b_1^2b_2)z^5$$
$$+ 8\,.\,7(3b_1b_2^2 + 3b_1^2b_3)z^6 + \ldots,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\therefore x = z + b_1z^2 + \left(b_2 + \frac{4}{2!}b_1^2\right)z^3 + \left(b_3 + \frac{5}{2!}\,.\,2b_1b_2 + \frac{5\,.\,6}{3!}b_1^3\right)z^4$$
$$+ \left[b_4 + \frac{6}{2!}(b_2^2 + 2b_1b_3) + \frac{6\,.\,7}{3!}\,.\,3b_1^2b_2 + \frac{6\,.\,7\,.\,8}{4!}b_1^4\right]z^5$$
$$+ \quad\cdot\quad\cdot\quad\cdot$$
$$+ \left[b_{n-2} + \frac{n}{2!}(2b_1b_{n-3} + 2b_2b_{n-4} + \ldots)\right.$$
$$+ \frac{n^{2|1}}{3!}(3b_1^2b_{n-4} + 6b_1b_2b_{n-4} + \ldots)$$
$$\left. + \frac{n^{3|1}}{4!}(4b_1^3b_{n-5} + 12b_1^2b_2b_{n-6} + \ldots) + \ldots\right]z^{n-1}$$
$$+ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\;;$$

wherein

$$n^{r|1} = n(n + 1)(n + 2)\ \ldots\ (n + r - 1),^{*}$$

and the parenthesis that $\frac{n^{r|1}}{(r + 1)!}$ multiplies, is

$$\Sigma\frac{(p + q + \ldots)!}{p!\,q!\ldots}b_i^pb_j^q\ \ldots,$$

in which

$$p + q + \ldots = r + 1, \quad\text{and}\quad pi + qj + \ldots = n - 2;$$

---

[*] In accordance with the notation

$$n^{r|d} = n(n + d)(n + 2d)\ \ldots\ [n + (r - 1)d].$$

that is to say, the *order* of this parenthesis in the letters $b_1, b_2, \ldots$ is $r+1$, its *weight* is $n-2$, and the numerical coefficients are those of the polynomial theorem.

In terms of the original letters $y, a_0, a_1, \ldots$ the series may be written in the more homogeneous form

$$x = \frac{y}{a_0} + \frac{y^2}{a_0^3}(-a_1) + \frac{y^3}{a_0^5}\left(-a_0a_2 + \frac{4}{2!}a_1^2\right)$$

$$+ \frac{y^4}{a_0^7}\left(-a_0^2a_3 + \frac{5}{2!}\cdot 2a_0a_1a_2 - \frac{5^{2|1}}{3!}a_1^3\right)$$

$$+ \frac{y^5}{a_0^9}\left[-a_0^3a_4 + \frac{6}{2!}a_0^2(a_2^2 + 2a_1a_3) - \frac{6^{2|1}}{3!}a_0(3a_1^2a_2) + \frac{6^{3|1}}{4!}a_1^4\right]$$

$$+ \quad . \quad . \quad .$$

$$+ \frac{y^{n-1}}{a_0^{2n-3}}\left[-a_0^{n-3}a_{n-2} + \frac{n}{2!}a_0^{n-4}(2a_1a_{n-3} + \ldots)\right.$$

$$\left. - \frac{n^{2|1}}{3!}a_0^{n-5}(3a_1^2a_{n-4} + \ldots) + \ldots + \frac{n^{n-3|1}}{(n-2)!}(-a_1)^{n-2}\right]$$

$$+ \quad . \quad . \quad . \quad .$$

It will be noticed that the coefficient of $\frac{y^{n-1}}{a_0^{2n-3}}$ is now a homogeneous function of $a_0, a_1, \ldots,$ of order $n-2$, weight $n-2$; and that the "polynomial coefficients" involved in such terms as $\frac{n^{3|1}}{4!}a_0^{n-6}(12a_1^2a_2a_{n-6} + \ldots)$ are to be chosen without reference to the exponent of a_0; while the latter exponent is related to the outside coefficient $\frac{n^{3|1}}{4!}$, by an obvious rule.

If $a_0 = 0$, let $a_{m-1}x^m$ be the first term in the given series, then the relation between x and y may be written in the form $x = f[z + \phi(x)]$, where $z = \frac{y}{a_{m-1}}$, and $f(z) = z^{\frac{1}{m}}$; hence Laplace's theorem gives a development for x that can be arranged in powers of $z^{\frac{1}{m}}$. As the general term now involves both n and m, the law is not so simple as in the case above, for which $m = 1$.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, March 3, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Mr. James Robert Baldwin, Davenport, Iowa; Mr. John Kelly Ellwood, Pittsburgh, Pa.; Miss Fanny R. M. Hitchcock, Philadelphia, Pa.; Professor Edward Anson Partridge, Central Manual Training School, Philadelphia, Pa.; Professor John Holmes Rand, Bates College, Lewiston, Me.; Professor Dr. Victor Schlegel, Trade School, Hagen, Westphalia, Germany; Professor James B. Shaw, Jr., Illinois College, Jacksonville, Ill.; Professor Herbert C. Whitaker, Central Manual Training School, Philadelphia, Pa.

Dr. I. J. Schwatt read a paper entitled "Application of Brocard's theories to conic sections." After a short discussion of Brocard's theories and the resulting properties, their relation was shown to the conic sections, especially to the equilateral hyperbola whose asymptotes are Simpson's lines.

IT is proposed that the NEW YORK MATHEMATICAL SOCIETY organize a general session, extending over several days, to be held annually during the summer vacation, at some appropriate place and time.

Such a session will be held this year in Brooklyn, on August 20, 21, 22, the days immediately preceding the session of the American Association for the Advancement of Science. At this time and place meetings of several national scientific societies, among them the American Geological Society and the American Botanical Society, will be held.

THE Council of the NEW YORK MATHEMATICAL SOCIETY has been considering with great care the present organization of the Society. It has decided to recommend to the Society, among other things, that its name be changed to the "AMERICAN MATHEMATICAL SOCIETY," that the Council be somewhat enlarged and divided into classes one of which shall be retired each year, that there shall be delivered before the Society a series of presidential addresses, and that provision be made for occasional meetings of the Society as a whole in other cities than New York. T. S. F.

HARVARD UNIVERSITY.—The scheme of instruction announced last June (see BULLETIN, vol. 2, p. 212) has been carried out during the present academic year without change except that one of the two reading courses has not been given. For the year 1894–95 undergraduate courses will be given in which the class-room work will amount to eighteen hours a week throughout the year. Besides this the following more advanced courses are offered: By Professor J. M. Peirce: Algebraic plane curves (first course); Quaternions (second course); Algebra of logic.† By Professor Byerly: Modern geometry; Differential and integral calculus (second course); Rigid dynamics.† By Professors Byerly and B. O. Peirce: Trigonometric series, spherical harmonics and potential function. By Professor B. O. Peirce: Wave motion. By Professor Osgood: Theory of functions (second course). By Dr. Bôcher: Theory of equations, algebraic analysis, invariants and covariants; Theory of functions (first course); Curvilinear co-ordinates and Lamé's functions.

All of the twelve courses just named will extend throughout the year; and each except the two marked † will consist of three lectures a week. The following three courses in which the work will consist of reading and investigation to be carried on by the students under the direction of the instructor are also offered: By Professor J. M. Peirce: Linear associative algebra. By Professor Osgood: Klein's icosahedron. By Dr. Bôcher: Laplace's equation. M. BÔ.

EMIL WEYR, professor of mathematics at the University of Vienna, died on January 25, aged forty-six years.

M. A. STERN, formerly professor of mathematics at the University of Göttingen, died in Zurich, on January 31, eighty-seven years old. T. S. F.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Antomari (X.). Application de la méthode cinématique à l'étude des surfaces réglées. Mouvement d'un corps solide assujetti à cinq conditions. [Thèse.] Paris, Nony, 1894. 4to. 113 pp. Illustrated.

Bunkofer (W.). Vectorenquadrate im ebenen, stetigen Gebilde. [Progr.] Wertheim, 1893. 4to. 21 pp. and 1 plate.

Daug (H. T.). Differential- och Integral-Kalkylens Användning vid Undersökning af Linier i Rymden och bugtiga Ytor. Del II. Upsala, 1894. 8vo. 178 pp. Mk. 6.20

Galán (G.). Estudio analítico-geométrico de las funciones hiperbólicas. Madrid, 1893. 4to. 60 pp. Mk. 2.50

—— Notas para el estudio del triángulo infinitesimal. Madrid, 1893. 4to. 65 pp. Mk. 2.50

Müller (K. E.). Untersuchungen über die Bruns'schen Integralgleichungen und über die Normalform der algebraischen Integrale algebraischer Differentialgleichungssysteme. [Diss.] Heidelberg, 1892. 8vo. 4 and 45 pp.

Rayet (G.). Notice sur la vie et les traveaux de J. J. B. Abria. Bordeaux, 1893. 8vo. 34 pp. Fr. 2.50

Riess (G.). Die singulären eindeutigen Ebenen- und Raum-Transformationen niederer Ordnung. [Diss.] Erlangen, 1893. 8vo. 63 pp.

Ritgen (A.). Untersuchungen über Ringschnitte. [Progr.] Schlettstadt, 1893. 4to. 66 pp. and 3 plates.

Serret (J. A.). Cours de calcul différentiel et intégral. 4me édition, augmentée d'une note sur la théorie des fonctions elliptiques par C. Hermite. 2 volumes. Paris, 1894. 8vo. 631 and 917 pp. Fr. 26.00

Taber (H.). On orthogonal substitutions that can be expressed as a function of a single alternate (skew-symmetric) linear substitution. 4to. (Am. Jour. of Math., vol. 16, pp. 123–130.)

II. ELEMENTARY MATHEMATICS.

Andriani (A.). Elementi di geometria Euclidea. Napoli, 1894. 8vo. Illustrated. Fr. 6.50

Arzelà (C.) ed **Ingrami** (G.). Aritmetica razionale. Bologna, 1894. 8vo. Fr. 3.50

Blanche (V. L.). Trigonométrie rectiligne et ses applications à l'arpentage. 6me édition. Huy, 1893. 8vo. 96 pp. Fr. 2.50

Gascó (L. G.). Tablas de logaritmos, cologaritmos y antilogaritmos de los números naturales y trigonométricos con los logaritmos de Gauss y de Mendoza, dispuestas de un modo nuevo. 5ta edición. Valencia, 1893. 8vo. 20 and 179 pp. Mk. 4.50

Hall (H. S.) and **Stevens** (F. H.). A text-book of Euclid's elements. Books II. and III. London and New York, Macmillan, 1894. 12mo. 140 pp. 2s.

Holzmüller (G.). Methodisches Lehrbuch der Elementar-Mathematik (im engsten Anschluss an die neuen Lehrpläne). (In 2 Theilen). Theil I. Leipzig, Teubner, 1894. 8vo. 8 and 212 pp. Illustrated. Cloth. Mk. 2.40

Pichot (J.). Éléments d'arithmétique. 3me édition. Paris, Hachette, 1894. 8vo. 258 pp. Fr. 3.00

Richardson (G.) and **Ramsey** (A. S.). Modern plane geometry: being the proofs of the theorems in the syllabus of modern plane geometry issued by the Association for the Improvement of Geometrical Teaching, with the sanction of the Council of the A. I. G. T. London and New York, Macmillan, 1894. 12mo. 202 pp. Cloth. $1.00

Ritt (G.). Problèmes d'algèbre et exercices de calcul algébrique, avec les solutions. 13me édition. Paris, 1894. 8vo. 400 pp. Fr. 5.50

——— Problèmes de géométrie et de trigonométrie, avec la méthode à suivre pour la résolution. 9me édition. Paris, 1894. 8vo. 400 pp. Fr. 5.50

Schotten (H.). Inhalt und Methode des planimetrischen Unterrichts. Eine vergleichende Planimetrie. (In 3 Bänden.) Band II. Leipzig, Teubner, 1893. 8vo. 4 and 410 pp. Mk. 8.00

Serret (J. A.). Trattato di trigonometria. Versione da L. Fenoglio. Torino, 1893. 8vo. Illustrated. Fr. 3.50

III. APPLIED MATHEMATICS.

Albrecht (T.). Formeln und Hülfstafeln für geographische Ortsbestimmungen nebst kurzer Anleitung zur Ausführung derselben. 3te, umgearbeitete und sehr erweiterte Auflage. Leipzig, 1894. 8vo. Mk. 16.00

Boccardo (E. C.). Trattato elementare completo di geometria pratica. Parte II: Topografia. Torino, 1893. 8vo. With plates.

Bowditch (N.). The new American practical navigator. Revised by order of the Bureau of Navigation. Washington, 1893. 8vo.

——— Useful tables from the American practical navigator. Washington, 1893. 8vo. 11 and 433 pp.

Briggs (W.) and **Bryan** (G. H.). An elementary text-book of mechanics. Book I: Dynamics. London, Clive, 1894. 12mo. 192 pp. (University Tutorial Series.) 2s.

Busch (O. A.). Ströftag i Världsrymden. Nagra Stjernstudier. Stockholm, 1893. 8vo. 315 pp. 2 plates. Mk. 4.50

Englisch (E.). Thermoelektrische Untersuchungen. [Diss.] Tübingen, 1893. 8vo. 21 pp. with 1 plate.

Escary (J.). Mémoire sur le problème des trois corps. 2me édition. Foix, 1893. 8vo. 4 and 89 pp. Fr. 6.00

Grossmann (L.). Compendium der praktischen Volkswirthschaft und ihrer mathematischen Disciplinen. Theil II. Wien, 1894. 8vo. 80 pp. Mk. 5.00

Halboth (W.). Anwendung der Tangential- und Schmiegungs-Bewegung auf einige Bewegungen des Punktes. [Progr.] Augsburg, 1893. 8vo. 33 pp.

Hatt (P.). Des coordonnées rectangulaires et de leur emploi dans les calculs de triangulation. Paris, 1893. 4to. 7 and 190 pp. Illustrated. Fr. 6.00

Kirchhoff (G.). Vorlesungen über mathematische Physik. Band IV: Theorie der Wärme. Herausgegeben von M. Planck. Leipzig, Teubner, 1894. 8vo. 10 and 210 pp. Illustrated. Mk. 8.00

Langley (S. P.). The internal work of the wind. Washington, Smithsonian Institution, 1893. 4to. 3 and 23 pp. (Smithsonian Contributions to Knowledge, No. 884.) $0.25

Müller (J.). Lehrbuch der kosmischen Physik. 5te umgearbeitete und vermehrte Auflage von C. F. W. Peters. (Ergänzungsband zu sämmtlichen Auflagen von Müller-Pouillet's Lehrbuch der Physik.) Braunschweig, Vieweg, 1894. 8vo. 23 and 908 pp. Illustrated. With 25 tables and an atlas of 66 plates in 4to. Mk. 26.00

Neumann (F.). Vorlesungen über mathematische Physik, gehalten an der Universität Königsberg. Heft VII: Vorlesungen über die Theorie der Capillarität, herausgegeben von A. Wangerin. Leipzig, Teubner, 1894. 8vo. 10 and 234 pp. Illustrated. Mk. 8.00

Pionchon (J.). Electricité industrielle. Cours fondé par la Société des amis de l'Université. 1re année (1893-1894). Leçons sur les notions fondamentales relatives à l'étude et à la mesure de l'énergie électrique. Bordeaux, Laurens, 1894. 8vo. Première livraison. 18 pp. Fr. 0.60

Preston (T.). The theory of heat. London and New York, Macmillan, 1894. 8vo. 730 pp. Cloth. $5.50

Quiquet (A.). Représentations algébriques des tables de survie ; généralisation des lois de Gompertz, de Makeham, etc. 2me édition. Paris, 1803. 8vo. 60 pp. Fr. 4.50

Résal (J.). Ponts métalliques. Vol. I. : Calcul des pièces prismatiques, renseignements pratiques, formules usuelles, poutres droites à travées indépendantes, ponts suspendus, ponts en arc. 2me édition, revue et augmentée. Paris, Baudry, 1893. 8vo. 41 and 642 pp. Complete in two volumes. Fr. 40.00

Ruotolo (R.). Corso di topografia e sue applicazioni. Vol. II: Altimetria. Omnimetria ordinaria. Celerimetria ed applicazioni della topografia, seguite da tutte le istruzioni ministeriali per la formazione del Catasto Italiano e da diverse appendici. Napoli, 1893. 8vo. Illustrated. Fr. 18.50

Salviati (E.). Elementi di astronomia nautica. 2da edizione. Genova, 1893. 8vo. With an atlas of 7 plates. Fr. 8.50

Wiedeburg (O.). Ueber die Gesetze der galvanischen Polarisation und der Elektrolyse. [Habilitationsschrift.] Leipzig, 1893. 8vo. 46 pp.

UTILITY OF QUATERNIONS IN PHYSICS.

Utility of Quaternions in Physics. By A. McAULAY, M.A., Lecturer in Mathematics and Physics in the University of Tasmania. New York, Macmillan & Co., 1893. 8vo, xiv and 108 pp.

THIS publication is an essay that was sent, in December, 1887, to compete for the Smith's Prizes at the University of Cambridge. It is now published in order to show the advantages of quaternion methods in the treatment of mathematical physics. The work deals with the usual theories of elastic solids, electricity, magnetism, hydrodynamics, and vortex motion. Some advances in these theories have also been made, such as: the expression of stress in terms of strain in the most general case of finite strain in an æolotropic body, including variations of temperature; the most general mechanical results of Maxwell's theory of electrostatics and its equivalent stress; a new equation in vortex motion by means of which the author has endeavored to deduce certain general phenomena that would be exhibited by vortex atoms acting upon one another, and from which he concludes that Hicks's vortex-atom theory is more promising than Sir William Thomson's in respect to a probable explanation of the fundamental facts of gravitation, inertia, etc.

The Introduction is devoted to a discussion of the advantages of quaternions over analytical geometry, and a statement in analytic form of the principal new results of the book. The author says, p. 2: "I believe that physics would advance with both more rapid and surer strides were quaternions introduced to serious study, to the almost total exclusion of Cartesian geometry, except in an insignificant way as a particular case of the former." And again, p. 3: "If only on account of the extreme simplicity of quaternion notation, large advances in the department of physics now indicated are to be expected. Expressions which are far too cumbrous to be of much use in the Cartesian shape become so simple when translated into quaternions that they admit of easy interpretation, and what is perhaps of more importance, of easy manipulation." To emphasize this remark, the author then compares the quaternion equation

$$\rho_1' S\nabla_1 \Psi \reflectbox{D} w \Delta = 0$$

with its exact Cartesian equivalent. One of the three equations that are required for this equivalent occupies six lines of print, from which the other two may be derived by attention to symmetry! [Thomson and Tait, p. 463, eq. (7).]

The above equation embodies several features of notation that appear frequently throughout the book. First, the suffixes, that point out corresponding operator and operand, and which are to be omitted after the differentiations are performed. That this notation is convenient is seen by the compact form in which it puts Leibnitz's theorem, viz., $D^n uv = (D_1 + D_2)^n u_1 v_2$. Next is the inversion of the usual order of operator and operand, ρ' being the operand, and ∇ the vector differential operator $i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}$. The inverted nabla ($\nabla$) at the end is another instance of this violation of the usual custom, since it operates on all the variables of the term in which it occurs. The desirability of these perversions of custom is, of course, greater in quaternions than in analytics, on account of the non-commutative property of quaternion multiplication; but I call to mind an instance in Johnson's Differential Equations, p. 162, where an fD is used in Mr. McAulay's sense of f_1D_1, the function f being a determinant with D's in the last column.

The last and most important feature of this equation is the new operator, $_{\Psi}\mathfrak{q}$, that Mr. McAulay has introduced into quaternions, and which contributes greatly to the power of that analysis in physics. It is an operator formed from the linear-vector function Ψ by changing its coefficients into differential symbols with respect to those coefficients, after the manner of forming operators from quantics in algebra. The following considerations show that the ordinary symbol ∇ is more appropriate for this operator than the new symbol $\mathfrak{q}$:

Let Q, which is a function of the nine coefficients of Ψ, be expressed as a function of the three independent vectors Ψi, Ψj, Ψk, where i, j, k are any given rectangular unit vectors. Then the ordinary partial operator, $_{\Psi i}\nabla$, is a *linear* function of i, and it may therefore be appropriately and conveniently written in the form $_{\Psi}\nabla i$. The double operative symbol found by dropping the i is exactly Mr. McAulay's new operator. Since the total variation of Q is the sum of its partial variations, we have

$$\delta Q = -S\delta\Psi i\,_{\Psi}\nabla i\,.\,Q - \text{etc.} = -S\delta\Psi\zeta\,_{\Psi}\nabla\zeta\,.\,Q.$$

Mr. McAulay does not mention the conjugate property $Si\,_{\Psi}\nabla j = Sj\,_{\Psi}\nabla i$, etc., although it is used in his proofs, *e.g.*, in finding 15*a*, p. 32.

When Ψ is self-conjugate, the vectors Ψi, Ψj, Ψk are not independent. This means that Q may be expressed in various ways as a function of these vectors without altering its actual value, so that $_{\Psi}\nabla$ is indeterminate unless we make some supposition as to the form of the function. This we may do by assuming it as symmetric in $Si\Psi j$ and $Sj\Psi i$, etc. We then have the operator for the case of Ψ self-conjugate that Mr. McAulay defines, and which he says has "a slightly different meaning" from the general symbol. When Ψ is not self-conjugate, then $\frac{1}{2}({}_{\psi}\nabla + {}_{\psi'}\nabla)i = {}_{\bar{\psi}}\nabla i$, and $\frac{1}{2}({}_{\psi}\nabla - {}_{\psi'}\nabla)i = {}_{V.\epsilon}\nabla i = \frac{1}{2} V._{\epsilon}\nabla i$, where $\overline{\Psi} = \frac{1}{2}(\Psi + \Psi')$, and $V.\epsilon = \frac{1}{2}(\Psi - \Psi')$.

I may remark here, in connection with linear-vector functions, that Mr. McAulay's suggestion of the name *Hamiltonian* for them does not seem so appropriate as the algebraic term *nonion* or the geometric term *linear strain*. Hamilton's name is connected with the whole subject, and not with one part more than with another. It is better to distinguish what others have introduced into the subject than what Hamilton has done, and it would be appropriate, in carrying out this idea, to call the operator ${}_{\psi}\nabla$ a "McAulayan."

The symbolic vectors ζ, ζ that have been used to express the value of δQ above, and which Prof. Tait pronounces purely and entirely Hamiltonian, are also freely used throughout the book. One of these vectors stands for ∇, the other for the variable ρ upon which it operates. Thus, $\sigma = -\zeta S\zeta\sigma$, and $Q(\zeta, \zeta) = Q(i, i) + Q(j, j) + Q(k, k)$, where Q is a linear function in each of its variables. In general, if $\phi\omega = -\Sigma\beta S\alpha\omega$, then $Q(\zeta, \phi\zeta) = \Sigma Q(\alpha, \beta)$. These and other formulas involving this symbolic vector furnish valuable suggestions as to important transformations in subsequent work.

Equations (8), (9) on page 19 are generalizations of what Tait and Hicks have proved, and deserve attention on account of their analogies. The extension of Mr. McAulay consists in making Q a sedenion (linear-quaternion function) instead of a quaternion—an extension that is readily made on account of the common distributive properties of the two numbers. Instead of the proof by infinitesimals, one may regard Q as at first a linear function of its position, when the theorems are obvious for finite rectangular figures. The extension to Q any continuous function and for any figures follows at once by the usual method of limits, *i.e.*, that method which regards the actual distribution of Q as the limit of a distribution which is linear throughout small rectangular elements of volume, the form of the linear function changing as one passes from each element of volume to an adjacent element.

These theorems are analogous to the well-known theorem for integration over a line AB, viz., $\int_A^B dQ = Q_B - Q_A$. Calling dQ a perfect line differential, and $Q(Vd\Sigma\nabla)$, $Q(\nabla)ds$, perfect surface and volume differentials of Q [∇ operating on Q], then all three theorems are included in the statement that a perfect differential taken over a given figure equals its corresponding integral (into an element of boundary) taken over the boundary of the figure.

The importance of these theorems may be inferred from the fact that they include the theorems of Green and Stokes as particular cases. Mr. McAulay has put them in compact forms that are easily remembered and applied. The advantage of ∇ and the full quaternion methods over any half-hearted attempts at quaternions in the way of vector analysis and *Curl* operators, is well illustrated in the case of these theorems.

In the subject of potentials, it is shown that Poisson's equation holds for a quaternion distribution, *i.e.*, that $\nabla^2 q = 4\pi Q$, where $q = \iiint u Q ds$, u being the reciprocal of the distance between a given point and any point of the volume.

Passing to the subject of elastic solids we find ample illustration of the superiority of quaternions. If every point of a body receive a displacement η, then $\chi\omega = \omega - S\omega\nabla . \eta$ is the radius vector of an ellipsoid into which the radius vector ω of a unit sphere would strain by the displacement. This unit sphere is, of course, the magnified image of an indefinitely small sphere at the point considered. The author takes $\chi\omega = q\psi\omega q^{-1}$, where ψ represents the pure strain and $q(\ )q^{-1}$ the subsequent rotation. The value of this rotation when the strain is small is incorrectly given (p. 26) as $V\theta(\ )$; it should be $V(1+\theta)(\ )$.

The stress function is ϕ, i.e., ϕi is the stress upon unit of surface normal to i. It is a tension or pressure according as the angle between i and ϕi is acute or obtuse. The stress upon the surface $d\Sigma$ is therefore $\phi d\Sigma$. By the theorems of differentiation already referred to, the sum of all such forces acting upon a closed surface of the body is the volume differential $\phi\nabla ds$ taken throughout the enclosure; hence $\phi\nabla$ is the *force* per unit volume on the element ds.

More generally $V\rho\phi d\Sigma$ is the vector moment of $\phi d\Sigma$ about the origin, and the sum of these moments for all points of the closed surface equals the volume differential, $V\rho\phi\nabla ds$, taken throughout the enclosure. Considering the partial variations of ρ and ϕ separately, this is a moment per unit volume of $V\rho_1\phi\nabla_1 + V\rho\phi_1\nabla_1$. The last term is that due to the force $\phi\nabla$ already found; hence $V\rho_1\phi\nabla_1$, ($= V\zeta\phi\zeta = 2\epsilon$, where $V . \epsilon$ is the rotary part of ϕ), is the *couple* per unit volume on the element ds.

These preliminaries being established with respect to a body subject to a strain χ, due to a displacement η, and having at every point a stress ϕ, the author now gives to every point, $\rho' = \rho + \eta$, a small additional displacement, $\delta\eta$. It is assumed that the potential energy of the body is the sum of the potential energies of its elements, $= \iiint w ds_0$, where w is the potential energy per unit of unstrained volume. Computing δw from this and from the work done by stresses on the surface and throughout the enclosure, $\delta\omega$ comes out a function of $\delta\eta$ and $\overline{\phi}$ (the pure part of ϕ) only In other words, the potential energy is independent of the stress couples. It is similarly found, by expressing $\delta\eta$ in terms of the strain, to be independent of the rotary part of the strain, $q(\ )q^{-1}$, viz., $\delta w = -mS\delta\psi \,.\, \psi^{-1}\zeta\varpi\zeta$, where $m = ds/ds_0$, the volume modulus of the strain, and ϖ is the pure stress due to the pure strain ψ. It is also shown that the rotation $q(\ )q^{-1}$ merely rotates the stress along with it, or that $\overline{\phi}\omega = q\varpi(q^{-1}\omega q)q^{-1}$.

Thus, finally, w is a function of the pure strain ψ only, so that $\delta w = -S\delta\psi\zeta\, {}_{\psi}\nabla\zeta \,.\, w$. Comparing this with the preceding value of δw, remembering that $\delta\psi$ is arbitrary, we have

$$m(\varpi\psi^{-1} + \psi^{-1}\varpi) = 2{}_{\psi}\nabla w.$$

This equation is explicitly solved by Mr. McAulay, giving ϖ in terms of ψ and w, which are supposed known. A simpler solution is also obtained by taking w as a function of $\psi^2 = \Psi$, transforming the value of δw so found back in terms of $\delta\eta$, and comparing with the value of δw already found in the same terms. This gives, directly,

$$m\overline{\phi}\omega = 2\chi\, {}_{\Psi}\nabla w\chi'\omega = 2\rho_1{}'S\rho_2{}'\omega S\nabla_1\, {}_{\psi}\nabla w\nabla_2 .$$

One of the three Cartesian equivalents of this equation is given in the Introduction, where it occupies ten lines of print.

The author then finds the general equations of internal equilibrium. The particular case for no bodily forces is given in Thomson & Tait, p. 463, eq. (7). [See the first equation in this review.]

In considering variations of temperature, Mr. McAulay states that the above results are still true, w being then a function of the temperature as well as a function of ψ. In treating the thermodynamics of the subject, an entropy function, f, is found by application of the second law, which is of the same form as for gases, viz., $\delta f = \delta H/t$.

Next comes the case of small strains, when the various general processes are much simplified. The results for small strains are, of course, well known in their Cartesian form, but the author adds, "it cannot be bias that makes these quaternion proofs appear so much more natural and therefore more simple and beautiful than the ordinary ones."

The general problem of isotropic bodies, the transformation to orthogonal co-ordinates, which is easier in quaternions than in Cartesian geometry, Saint-Venant's torsion problem, and a general treatment of wires, complete the chapter on elastic solids.

As an illustration of the general character and scope of the work this synopsis is perhaps sufficient, but I cannot resist adding a brief outline of the author's treatment of the general problem of electrostatics in the section on electricity and magnetism.

He considers that Maxwell starts with space uniformly filled with a substance called electricity, whose resultant displacement, **D**, at any point is linearly connected with the electromotive force, **E**, at that point, i.e., $\mathbf{D} = K\mathbf{E}/4\pi$, where the nonion K depends upon the state of the medium at the point considered. It is further assumed that this displacement and force bear to each other the ordinary energy relation between work, force, and displacement; i.e., if w be the potential energy per unit volume, then $\delta w = -S\mathbf{E}\delta\mathbf{D}$. This, by the principle of conservation of energy, may be integrated on the supposition that the displacement, and therefore the force, has increased uniformly from 0 to its given value. This gives $w = -S\mathbf{DE}/2$; from this, and the preceding value of δw, he finds $\delta w = -S\mathbf{D}\delta\mathbf{E}$, and hence that K is self-conjugate.

As a further consequence of the conservation of energy, the line integral of **E** round a closed curve is zero, i.e., $\mathbf{E} = -\nabla v$.

The charge in any space is the surface integral of the displacement outwards. Thus, if σ, D, are surface and volume densities, $\sigma = SUd\Sigma_a\mathbf{D}_a + SUd\Sigma_b\mathbf{D}_b$, a, b denoting the two faces of the element; also $D = -S\nabla\mathbf{D}$.

Two values of $W = \iiint w\,ds$ are then found, one directly by substituting the value of w given above, and the second by substituting in the first $\mathbf{E} = -\nabla v$ and changing the perfect volume differential into a surface differential. From twice the second, minus the first, a third value of W is obtained which is the one used by the author in subsequent work.

The author next, following the example of Helmholtz in the particular case of K a scalar, gives to every point of space a small displacement $\delta\eta$, vanishing at infinity, and finds the consequent increment δW in W. He then, in order to find the equivalent stress $\phi\omega$, puts δW in the form $-\iiint S\delta\eta_1\phi\nabla_1 ds$. The value of ϕ thus found is

$$\phi\omega = -V\mathbf{D}\omega\mathbf{E}/2 - {}_{\psi}\nabla\omega \,.\, w.$$

The latter part of this stress is unknown, depending upon the unknown relation between ψ, the pure strain of the medium, and w. Its value is found by Mr. McAulay under one or two assumptions as to the relations of these quantities.

When ω is in the plane of **D**, **E**, the first part represents the vector ω turned through 180° round the bisector of **D** and **E** as axis, and lengthened in the ratio of $T\mathbf{DE} : 2$. When ω is perpendicular to **D**, **E**, the first part represents ω altered in the ratio $S\mathbf{DE} : 2$.

In the ordinary case of air, $K = 1$, and $\phi\omega = -\mathbf{E}\omega\mathbf{E}/8\pi$ = vector ω turned 180° round **E** as axis, and lengthened in the ratio $T\mathbf{E}^2 : 8\pi$. This is Maxwell's result (vol. I, chap. V), as may be seen by taking ω alternately parallel and perpendicular to **E**. This result is, of course, immediately obtained by following Maxwell's method in that chapter.

I can hardly make substantial addition to what Prof. Tait has said in praise of the book (*Nature*, Dec. 28, 1893). It is of undoubted scientific value, and the work of a man of genuine power and originality. I find fewer obscurities in the work than in many of the so-called elementary text-books upon the same subjects. It can hardly be doubted that this book will go far towards accomplishing the author's purpose of arousing serious interest in quaternion analysis.

With regard to the fancied difficulties of the subject, I can speak from experience with large classes of average students that it is as easily acquired as analytical geometry. When the fundamental ideas of quaternions have drifted by their weight into the elementary schools, so that students may attack the subject with minds properly prepared for it, I believe that quaternion methods will be found clearer and simpler to the average student than analytical methods. It should not be forgotten that all elementary text-books are built upon the lines of analytical geometry, and that the great need of quaternions at present is a carefully prepared series of text-books that lead up to and finally employ the highest quaternion analysis in all departments of mathematics and physics, to the exclusion of analytical geometry, except as a special case of the former. A. S. HATHAWAY.

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NOTE UPON THE HISTORY OF THE RULES OF CONVERGENCE IN THE EIGHTEENTH CENTURY.*

BY PROF. G. ENESTRÖM.

In the introduction to his interesting article "Evolution of Criteria of Convergence" (BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY, vol. 2, 1892, pp. 1–10), Professor Cajori mentioned several authors previous to the present century who had recognized the necessity of verifying the convergence of a series before making use of it in mathematical investigations. In this brief note I beg to call attention to two other mathematicians, Maclaurin and Stirling, who had considered rules of convergence; the first of these deserves, in my opinion, a signal place in the history of these rules.

In fact, the fundamental theorem of convergence attributed (l. c., p. 3) by Professor Cajori to Cauchy was established previously by Maclaurin in article 350 of his "Treatise on Fluxions" (Edinburgh, 1742). It is true that Maclaurin enunciated the rule in a geometrical form, but it is very easy to translate it into algebraic language. The fact that this rule is due to Maclaurin was pointed out, I believe, by Montucla, but it does not seem to have been sufficiently appreciated until quite recently. I called attention to it in the *Bibliotheca Mathematica*, 1886, p. 48, and it was mentioned later by R. Reiff in his "Geschichte der unendlichen Reihen" (Tübingen, 1889, p. 120), as well as by W. W. R. Ball in the *second* edition of his "Short Account of the History of Mathematics" (London, 1893, p. 394).

As regards Stirling, he demonstrated in his "Methodus Differentialis" (London, 1730) the following theorem: Let

$$W = w_0 \,.\, w_1 \,.\, w_2 \,.\, w_3 \ldots$$

and

$$w_s = \frac{1}{m}\,\frac{z^\theta + az^{\theta+1} + bz^{\theta+2} + \ldots}{z^\theta + cz^{\theta+1} + dz^{\theta+2} + \ldots},$$

then

$$W = 0, \text{ if } m > 1, \text{ or if } m = 1 \text{ and } a < c;$$

$$W = \text{a finite quantity} \gtrless 0, \text{ if } m = 1,\ a = c;$$

$$W = \infty, \text{ if } m < 1, \text{ or if } m = 1 \text{ and } a > c.$$

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* Translated by the Editors.

It is easily seen that this theorem plays the same rôle in the theory of infinite products that is taken by Gauss's rule of convergence, to which Professor Cajori refers on the second page of his note, in the theory of infinite series. Stirling attempted to make use of this theorem in order to establish the convergence of certain infinite series, but his efforts in this direction met with little success. More complete information in regard to Stirling's theorem and its applications to series are contained in my note "Ett konvergens kriterium från bôrjan of 1700-talet" (A rule of convergence of the beginning of the eighteenth century) inserted in the Bulletin (Öfversigt) of the Academy of Sciences of Stockholm, 1879, No. 9, pp. 71-84.

STOCKHOLM, *March* 31, 1894.

MEYER'S REPORT ON THE THEORY OF INVARIANTS.

Bericht über den gegenwärtigen Stand der Invariantentheorie von Dr. FRANZ MEYER, Professor a. d. kgl. Bergakademie, Clausthal. Contained in Vol. I. of the *Jahresbericht der deutschen Mathematiker-Vereinigung*, pp. 79-292. Berlin, Reimer, 1892. 8vo.

BOTH as regards its problems and as regards its methods, the theory of invariants presents a more heterogeneous mass of materials for the summarizer of its history to deal with than do most of the mathematical disciplines which have undergone so rapid a development during the second half of our century. Dr. Franz Meyer undertook, therefore, a less attractive task in preparing his "Bericht über den gegenwärtigen Stand der Invariantentheorie" than will fall to the lot of those who will assume similar labors for the German mathematical society in other fields. With infinite patience and comprehensive knowledge, Dr. Meyer has given us, in a volume of 200 pages, a condensed statement, not only of all the methods, but of a large part of the results, of research in the field of invariants, from its beginning in the discoveries of Boole, in 1841, to the very date of the preface to the present work, August, 1892. The period 1841-67 is, indeed, passed over more summarily than the last quarter-century, the memorable work done by Cayley, Sylvester, Hermite, Brioschi, and others in the earlier period being surveyed in a brief "Rückblick," though to some extent also discussed in the body of the report.

It is quite impossible to summarize in a few pages what is already in 200 pages a condensed summary; we shall therefore content ourselves with pointing out a few features of the Report. Dr. Meyer looks upon the matter—rather unfortunately, we think—as naturally and historically dividing itself into two main parts: the first having reference to the problem of equivalence, that is, the linear transformability of algebraic forms into each other, and the substitutions whereby these transformations are effected; the second to the algebraic relations between invariantive forms. The report is correspondingly divided into two parts, of which the first is naturally very much the smaller, occupying less than 30 pages. This first part would have been still more brief had the author confined himself to what we generally associate with the theory of invariants in connection with the question of equivalence. But the section on the equivalence-problem for quadratic and bilinear forms, and the section on the same problem for higher forms, are followed by a section entitled "Forms with linear transformations into themselves," relating to researches on the general problem of finding all possible groups of substitutions of n variables and the rational entire invariants of these groups. Here are mentioned in the first place the discoveries of Schwarz and Klein, which arose from researches in the theory of functions; then the connection with the theory of equations of the fifth degree, as developed by Klein in the "Ikosaeder"; then many other developments, connected with the transformation of elliptic functions, the theory of linear differential equations, etc. Of purely algebraic researches in this field, perhaps the most notable mentioned are Gordan's investigation of binary forms with linear transformations into themselves, Jordan's general discussion of finite groups of substitutions for n variables from the point of view of the theory of substitutions, and the generation of the "regular-polyhedron" equations given in Gordan's lectures on invariants.

The second part, which comprises the great body of the researches in the theory of invariants, consists, besides a few pages devoted to "Irrationelle Fragestellungen,"—which pages include the question of canonical forms and of the primitive forms corresponding to given covariants,—of three large divisions: "Endlichkeitsfragen," "Methodik," "Specielle Substitutionsgruppen und Formen." Of course the "questions of finiteness" are the questions relating to the constitution of the complete system of ground-forms pertaining to a given system of algebraic forms; and our author includes under this head not merely the question whether the system of ground-forms is finite or not, but also the methods of constructing the complete system, the question of the syzygies to which the ground-forms are subject, and the methods of denumeration, by which the types of the ground-forms and syzygies, and the number of each type, are determined without the actual construction of them. Thus the question of finiteness, strictly speaking, occupies only one of the five sections into which the division on "Endlichkeitsfragen" is subdivided. In this is sketched the beginning of the question in the early work of Cayley and Sylvester; then we have an outline of Gordan's proof that the complete system actually is finite in the case of any given system of binary forms, of subsequent improvements of this proof, and of Peano's extension of the proof to the case of systems of binary forms involving different pairs of variables, subject to independent linear substitutions. Of an entirely different kind are the proofs given by Mertens and Hilbert, some sixteen years later, depending, as they do, on the finiteness of the number of fundamental positive solutions of a system of linear diophantine equations; these proofs are vastly simpler than Gordan's, especially Hilbert's, which is so brief as almost to be designated as instantaneous; on the other hand, they do not, like Gordan's, present any process by which the complete system may actually be obtained; moreover, they go outside the natural domain of rationality of the problem. All these investigations relate to binary forms only; and it was not until 1890 that the finiteness of the complete systems for forms in n variables was demonstrated. This was done by Hilbert in his great work in vol. 36 of the *Mathematische Annalen*, the distinguishing feature of which (so far as it bears on our subject) is that it operates in a far wider field than that of invariants, and deduces the theorem of finiteness for invariants of forms in any number of variables—and even for the syzygies of every rank and for the number of such ranks—from far more general theorems. The Report gives an outline of Hilbert's proof; the proof has been simplified by Story (*Annalen*, vol. 41) since the appearance of the Report.

Passing over the intervening sections, we shall only mention further, in regard to the division on "Endlichkeitsfragen," that we have in the last section of this division ["Abzählende Richtung"] a very intelligible account of the Cayley-Sylvester methods of denumeration by means of generating functions; a mention of the remarkable results arrived at by Jordan for the superior limits of the degree and order of the covariants of a system of binary forms; and an account of the work of Capelli and Deruyts on the extension of the Cayley-Sylvester theorem for the number of linearly independent covariants of binary forms to the case of forms in several sets of n variables.

In the division on "Methodik," we have, under the subhead of "Symbolik und graphische Darstellung," not only an account of the great symbolic method initiated by Cayley, developed into a thoroughly organic system by Aronhold and Clebsch, and used with so much power by Gordan, but also a notice of the chemico-algebraic method of graphs devised by Sylvester and extended by Clifford, and of MacMahon's remarkable transformation of the question of seminvariants into a question of symmetric functions. Under the other heads of this division, and under the last division of the Report, that on "Specielle Substitutionsgruppen und Formen," the number of points that come up for treatment is so great that a continuation of even such cursory notice as we have been giving would be fatiguing. Suffice it to say, therefore, that whoever consults the Report will be impressed by the fact that the development of the Theory of Invariants in recent years, while overshadowed by the brilliant conquests made in the domain of the Theory of Functions, has been by no means at a standstill. Not to speak of the excursus into the field of differential invariants made by Sylvester and his followers, MacMahon, Hammond, and others, signal advances have been made in the central theory, especially by Capelli, Stroh, Study, and Deruyts. Dr. Meyer, in his preface, expresses regret that he found it impossible, except in a few instances, to include the geometrical applications of the theory in the scope of the Report. With this exception, the student of the theory of algebraic forms and invariants will find in the Report a remarkably full abstract of researches in this domain, accompanied by accurate bibliographical references, and will feel under great obligation for the assistance rendered by this result of Dr. Meyer's great learning and painstaking industry. It ought to be especially useful to any one undertaking to present, in a systematic work, the body of doctrine which is the outcome of the varied and often heterogeneous researches outlined in this compendious report.

F. FRANKLIN.

BALTIMORE, *April*, 1894.

CAJORI'S HISTORY OF MATHEMATICS.

A History of Mathematics. By F. CAJORI. New York, Macmillan & Co., 1894. 8vo, 14 and 422 pp.

IT is a long time since an American work has been awaited with so much anticipation by readers of mathematics as Professor Cajori's recent history. The book had been extensively advertised, there was and is a growing demand for such works, and the supply of material was well-nigh inexhaustible. But while few books have ever enjoyed such advantages, few books have ever so seriously failed to improve them. This is a harsh statement and should neither be lightly made nor lightly accepted. It is based upon the following facts, which are stated as concisely as an adverse criticism allows.

First. The work is, in very considerable measure, merely a paraphrase of portions of better works to be found in the libraries of most readers. Witness the following extracts. (Bracketed clauses are merely transposed.)

Cajori.	*Gow.*
"Plato was born at Athens in 429 B.C., the year of the great plague, and died in 348. He was a pupil and near friend of Socrates, but it was not from him that he acquired his taste for mathematics. After the death of Socrates, Plato travelled extensively. . . . He went to Egypt, then to Lower Italy and Sicily, where he came in contact with the Pythagoreans. Archytas of Tarentum and Timæus of Locri became his intimate friends." (p. 29.)	"Plato was born . . . at Athens in 429 B.C., the year of the great plague. [He died . . . in 348.] He was a pupil of Socrates, . . . but he did not derive from this teacher his enthusiasm for mathematics. . . . After the death of Socrates [Plato] went away from Athens. . . . He went certainly to Egypt . . . and lastly to Magna Græcia and Sicily, where he [consorted with Pythagoreans. . . . He] became a close friend of Archytas and Timæus of Locri." (p. 173.)
	Ball, 2d Edn.
"At the age of thirteen he is said to have been familiar with as many languages as he had lived years. About this time he came across a copy of Newton's *Universal Arithmetic*. After reading that he took up successively analytical geometry, the calculus, Newton's *Principia*, Laplace's *Mécanique Céleste*," etc. (p. 318.)	"When thirteen he was able to boast that he was familiar with as many languages as he had lived years. It was about this time that he came across a copy of Newton's Universal Arithmetic ; . . . he soon mastered the elements of the analytical geometry and the calculus. He next read the *Principia*, and . . . Laplace's Mécanique Céleste," etc. (p. 474.)
	Fink.
"This problem was reduced to another, now generally known as Malfatti's problem : to inscribe three circles in a triangle that (*sic*) each circle will be tangent to two sides of a (*sic*) triangle," etc. "Steiner gave without proof a construction, remarked that there were thirty-two solutions, generalised," etc. (p. 296.) (A reference is given to Fink in a preceding sentence, but not on the part transcribed.)	"Diese Aufgabe reduzierte er auf die jetzt allgemein als "*Malfatti'sches* Problem" bekannte Forderung, in ein gegebenes Dreieck drei Kreise so einzubeschreiben, dass jeder Kreis," etc. "Steiner gab (ohne Beweis) eine Konstruktion, führte an, dass es zweiunddreissig Lösungen gebe, und verallgemeinerte," etc. (p. 202.)

The above extracts are only specimens of many cases that might be cited, in which the author seems to have copied, without giving due credit, from Gow, Ball, Hankel, Cantor, and Suter, and apparently from Halsted's introduction to his monograph on Bolyai. Sometimes the extract is from one author and the credit is assigned to another, but usually no credit is given to any one. The first seventy-five pages are largely paraphrased from Gow; then Hankel is drawn upon; and, finally, Ball, Cantor, Suter, and a few other sources are utilized. Now and then a note refers to one of these writers, but it rarely happens that, at such times, the writer cited is followed more closely than on other occasions. One cannot but wonder why Professor Cajori did not, at the beginning, frankly say that he had copied *ad libitum* from three or four authors, instead of giving only occasional credit.

But it may be said, laying aside the ethics of the matter, that the work claims (which it does not) to be merely a compilation; that good authors have been selected, and their words carefully transcribed. A single selection may be given in reply to such a suggestion. This particular one is taken because, laying aside the seriousness of the discussion for a moment, it may cause a pardonable smile.

Cajori.	*Hankel.*
"Creditable work in theory of numbers and algebra was done by ***Fahri des Al Karhi***, who lived at the beginning of the eleventh century. His treatise is the greatest algebraic work of the Arabs. In it he appears as a disciple of Diophantus. He was the first to operate with higher roots and to solve equations of the form $x^{2n} + ax^n = b$." (p. 111.)	"Das grösste algebraische Werk der Araber, das wir besitzen, der Faḫrīdes Al Karḫī aus dem Anfange des 11. Jahrhunderts, welches ein genaues Studium des Diophant zeigt, geht über die ältere Algebra nur insofern hinaus als es auch mit höheren Wurzeln als Quadratwurzeln operiren und Gleichungen von der Form $x^{2p} \pm bx^p = \pm a$ auflösen lehrt." (p. 265.)

Fahri des Al Karhi! Does not every reader of the history of mathematics know that Fahri (or Al-Fakhrî) was the name of the *book* that Alkarkhî wrote? It is all explained on p. 245 of Hankel: "Einem Vezir Faḫr ul Mulk (†1017) eines anderen Buyiden Behă-ed-daula widmete Al Karḫī ein algebraisches Werk, *dem er zu Ehren seines Gönners den Namen Al Fahrī gab.*" A glance at Cantor (Vol. I, p. 655), or at Müller and his references, or at Heath (p. 24) would have saved the author from this most awkward blunder.

Second. The work is weak in bibliography, where it should be exceptionally strong. One has a right to expect a rich set of references to the standard literature of the day. Such references are offered by other histories, however humble, and every student needs them. Yet in this work there is not a single reference by volume and page. The bibliography is unscientific and meagre, and the use made of it may be called fictitious. It consists of a few standard histories, a few text-books, a few periodical articles not paged, and a number of works of no special value. As an example of the authorities cited, let any one who is familiar with the twenty-two volumes of the *Jahrbuch über die Fortschritte der Mathematik* already published (the last volume having over 1200 pages) consider such a reference to that work, without volume or page, as the one (p. 250) on the statement that Euler used π for 3.14159... in 1737! A reference to vol. 21, p. 30, or to the *Bibliotheca Mathematica* for 1889, p. 28, would have been of some value.

It is true that even the least of the works cited might, if referred to by volume and page, be useful to the student. But it is surprising that many works of so much more value than most of those mentioned are ignored. Some of these are easily accessible, like the valuable historical articles in the last edition of the *Britannica*, by Allman, Chrystal, Glaisher, Cayley, and Tait, or those in Smith's Dictionary of Biography, or those in Leslie Stephen's current work on National Biography. One misses all reference to the many biographical contributions by Battaglini, Cremona, Beltrami, Bertrand, Brioschi, de Comberousse, Darboux, Jacoli, and others, and to the numerous historical memoirs of Boncompagni, Cantor, Chasles, Curtze, Favaro, Friedlein, Tannery, Treutlein, Günther, de Haan, Henry, Mansion, Martin, Narducci, and Steinschneider, and to such common works, of much higher value than many named, as those of Unger, Sterner, La Cour, Heath, Rudio, Treutlein, Weissenborn, Wolf, Reiff, Woepcke, and to the recent contributions of Heiberg and Hultsch. These men are not unknown, nor are their works rarities. They have written extensively, and their contributions are valuable and are of our generation. Why should the only reference on Pascal, for example, be Madame Perier's Life, which was translated into English in 1744, while the valuable contributions by Cantor, Chrystal, Chasles, Todhunter, Wolf, Desboves, Pisko, Tannery, Henry, Bianco, to say nothing of Ball, Marie, Suter, Williamson, Hoefer, Montucla, Bossut, and others, are unmentioned? That the bibliography should omit the works of older writers like Kästner and Libri and Bossut, or a writer like Hoefer (of whom, however, Allman speaks with some favor), is not strange, but that such a sweeping omission is made is quite remarkable.

Third. But it may be said, and indeed it has been said, that this work is especially strong in relation to modern mathematics. While this will not excuse the errors in the treatment of the earlier development of the subject, to be mentioned hereafter, nor the weakness of the bibliography relating to that development, the claim should be considered. But however charitable the reader may be, he will close the final chapters with even greater disappointment than he experienced in reading the earlier ones. What, for example, does the work tell of the growth of the theory of substitutions and of groups? Say fourteen lines, all told. Might not one expect some mention of the contributions of Frobenius, Stickelberger, Kneser, Marggraf, Rudio, and our Professor Bolza, and, in general, a good résumé of the development of the subject? And on the theory of invariants, while the reader will find several helpful notes, might he not reasonably expect that a history of mathematics published in 1894 would give a fair condensation of Meyer's account in the first volume of the *Jahresbericht der deutschen Mathematiker-Vereinigung*, pp. 81–288? Even some reference to this elaborate memoir would have been of great value. And in the matter of theory of functions, if he were reading Forsyth, or Harkness and Morley, would he, after one or two attempts, ever go to this work again for any help in tracing the development of the subject? Or, to take a somewhat different illustration, if he were interested in the development of the considerable subject of mathematical tables from Herwart to our time, how little would he find in this work! A mere note, referring to Glaisher's monographs, would have been helpful. In the matter of modern biography, let the reader consider the two or three lines devoted to Sophus Lie, merely to say that he has applied finite continuous groups to the treatment of differential equations, and has helped to edit Abel's works. Much more is said of other living writers who are not worthy to unloose the latchet of Lie's shoe. Of "one of the most elegant contributions to differential geometry made in recent times," mentioned by Klein at Evanston, and indeed of the general scope of Lie's work, there is only what is contained in a dozen words.

It strikes an American pleasantly to see mentioned the names and labors of over thirty of his countrymen. While the number is disproportionate, and while thirty American mathematicians could not be found who would wish to be mentioned in a work which ignores the names of so many world-known promoters of the science, the effect in our own country may possibly be of value. The names or labors of from fifty to seventy-five men (not including contemporaries) who are much more entitled to mention than many who are given place, are wanting, while the selection of living mathematicians can scarcely be called a happy one.

Fourth. A final reason why the work is disappointing is apparent from the first page: the work is carelessly written. One who consults a history of any subject may reasonably expect to find the common facts of that subject, together with the names, dates, nationalities, and principal works of its leading contributors set forth in compact form. But when he reads of Metius without his value of π, of Nonius without mention of the nonius, of Napier's "analogies" and "rods" two pages after the discussion of Napier; when he finds the Christian name incorrectly given or frequently omitted; when he finds no dates assigned to a large number of writers, including men of the prominence of Galileo, Malfatti, Viviani, Bürgi, Cramer, and others equally well known,—may he not reasonably affirm that the greatest care has not been taken?

In the matter of the spelling of common oriental names, also, may not one rightly complain of the utterly unscientific policy pursued? It will be remembered that Hankel set forth (p. 225) a system of transliteration for such names, which he followed, but which has been accepted by no other leading writer on the history of mathematics. Among other things Hankel says that the symbol *ḫ* shall be sounded like the German hard *ch*, and he so uses it. Other writers generally use *ch* or *kh* for the same sound. But Professor Cajori, ignoring all recent authorities, ignoring the common dictionaries (v. *algorithm*), and ignoring Hankel's essential subscript, merely uses *h* as the equivalent of the German hard *ch*. The result is often curious. Consider, for example, p. 106. The author is evidently writing with Hankel (p. 260) and Cantor (p. 611) open before him. For some reason he intends to take Hankel's spelling of Muhammed ibn Mûsâ Alchwarizmî, viz.—Mohammed ben Mūsā al Ḫovārezmi. But in doing so he forgets the very important *al*, and creates a spelling that is *sui generis*. Even this is not so strange as the fact that, attempting to follow most writers, he derives *algorithm* from *Hovarezmi!* That it should come from Alchwarizmî, or from Hankel's *al Ḫovārezmi*, is clear enough, but the above etymology is certainly noteworthy.

Of the other errors in the book it is necessary to speak at no great length. A few may be mentioned to show that, like all first editions, the work is not free from them. In the bibliography, not considering matters of taste in the capitalization of German adjectives, errors will be found in numbers 38, 39, 47, 50, 55, 60, 65, 67, 76, 84, 88, 92, 95, 96, 97. The spellings of Deinostratus (p. 25) and Dinostratus (p. 32),—of Jakob Bernoulli (p. 182), Jacob and James (p. 236), and James (p. 237),—of Gand when Ghent is well Anglicized,—these are errors of taste rather than of fact. The Egyptian fractional symbol *ro* is spoken of as a dot; and while a dot was used by Ahmes, and is, for convenience, sometimes used in printing Egyptian fractions, it was never used as Professor Cajori states; a glance at Eisenlohr's table, or at the examples in the 2. Abschnitt, would show this. A further error in symbolism occurs in Diophantus' sign of subtraction, a strange error in view of the full discussion in Heath (p. 72), not to mention Gow (p. 109) and Cantor (p. 401). But with Heath the author seems unacquainted, else he would not err in his symbol for the first power of the unknown (so fully treated in Heath, pp. 57–69). A similar error appears in his Greek numerals, apparently from following Hankel too blindly. Such errors of titles, as of one of Plücker's works, or of spelling, as Midorge (p. 174), Tchirnhausen (index), Professor Moor (p. 330), etc., need scarcely be mentioned, since they are manifestly misprints. Similarly for such errors of the index as "Moivre, de, 245," and "De Moivre, 240," and the reference to p. 292 under "Substitutions," while the more important reference to p. 329 is wanting. Neither is it worth while to dwell upon the occasional specimens of poor English, illustrated by the use of "insolvability."

There are a number of errors in chronology, however, that may annoy the student. Such are the date of Tschirnhausen's birth and the date of Ivory's death, which latter comes from depending solely on Ball. In many cases about which there is much doubt the dates are given with apparent certainty, while in cases about which there is little or no doubt the dates are frequently omitted entirely or given with a question-mark. Thus, to Hipparchus and to others heretofore mentioned are assigned no dates, while to Archytas are assigned the dates 428–347, both of which are probably wrong, and to Pythagoras are assigned the dates "580 ?–500 ?", both of which are probably within a year of exactness. It is stated of Thales that he died in 546, which is quite doubtful, and in the case of Heron the Elder Gow's "c. 120," and the even more probable date c. 110, are ignored, and "c. 155" is assigned; in this latter the author apparently depends on Marie I, 177, forgetting that Marie is notoriously untrustworthy. The date 814 for the beginning of Al Mamun's reign is incorrect, as are others which it is impossible here to enumerate.

From this effort to call attention, fairly and without exaggeration, to the chief deficiencies in Professor Cajori's work, it should not be inferred that the book is without merit. Far from it. It tells in a popular way the general story of the growth of mathematics. It is well printed and is altogether an attractive piece of book-making. It is a pioneer in America and should be welcomed as furthering interest in a very important subject. Moreover, it makes an effort in the way of tracing the recent development of mathematics. For all this the work deserves credit, and Professor Cajori deserves thanks. But in view of what has been said, it seems only a plain statement of the truth to add that as a scientific treatise the work cannot be regarded as an authority.

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GRAVITATION AND ABSOLUTE UNITS OF FORCE.

ABSTRACT OF A PAPER READ BEFORE THE NEW YORK MATHEMATICAL SOCIETY AT THE MEETING OF APRIL 7, 1894.

BY PROF. W. WOOLSEY JOHNSON.

THE writer held that the conflict between gravitation and absolute units was irrepressible because of the impossibility of reconciling the practical necessities of the engineer with the scientific needs of the physicist. Accordingly most modern text-books admit both kinds of units. The history of the matter was briefly summarized. Weighing being the inevitable manner of comparing masses, the same terms have always been used to describe masses and the pressures produced by their gravitation. With the rise of mechanical science the conceptions of force and of mass must be differentiated. The older writers were content to write $P \propto mf$; force, mass, and acceleration might be expressed each in its own unit; but the modern method is to write such a relation in the form $P = kmf$, and, first establishing fixed units to be employed, to proceed to determine k. Since no occasion had hitherto arisen for a unit of mass as distinguished from a unit of weight, no difficulty was at first felt in adopting for m such a unit that $k = 1$, and hence $P = mf$, while the pound, the foot, and the second were the units of force, length, and time. In other words, in using $W = mg$ no inconvenience was felt from the fact that in assigning a numerical value to m its unit was not a mass weighing one pound, but a mass weighing g pounds. There would rarely be occasion to employ the numerical value of m, W/g being substituted for it in final results.

But g is found to be variable, and since our standards furnish us with an invariable mass, it is seen that we have been using a variable unit of force. The engineer and practical man, however, while admitting that mass and not force is the third primary unit, still finds it more convenient for his purpose to use this variable, or rather let us say 'local,' unit of force, in spite of the fact that in using the formula $W = mg$ this implies also a variable or 'local' unit of mass.

This variable unit of mass seems intolerable to a certain class of writers who object *in toto* to gravitation units. With these writers "the British unit of mass is the Imperial

Pound"; "the weight of a body is the force (a variable) with which the earth attracts the body"; and "the unit of force is the poundal, in which unit all forces must be expressed in order that $P = mf$ may be universally true." Accordingly "we ought to speak of the weight of a pound," and in examples we find a resistance specified as "204 lbs. weight" a force as "equal to the weight of a ton," etc. These writers of course put m for the number of pounds and bring out every force in poundals (or in dynes if C. G. S. units are used); on the other hand, those who freely admit gravitation units, while using $W = mg$, put W equal the number of pounds in dynamics just as most writers do in statics. The argument of the former class is this: In $W = mg$, W stands for weight and is a force, m stands for mass; surely m is the constant and W the variable, when g varies. Granted, say their opponents, that the mass is constant and the force variable. A force is the product of a numerical measure and a corresponding unit. If, in the case of a force varying with g, it be convenient (as it distinctly is) for certain purposes to throw this variableness upon the unit, and not upon the numerical measure, why not do so, as well as *vice versa* for other purposes which make the reverse process desirable?

The confusion which sometimes arises, and of which instances were quoted, is due to the double meaning not so much of the general term "weight" as of the special term "pound," now that it is customary to say that the pound is the unit of mass and to put m for mass. By an incautious use of language we are sometimes left in doubt as to whether a certain numerical value is to be assigned to m or to W.

In the opinion of the writer, those who employ gravitation units make a mistake in trying to erect the pound into the semblance of an absolute unit by referring to a conventional locality and to the illustration of a spring-balance graduated at such a locality. Thus Williamson says: "In this system the unit of force is the weight at some definite place [London] of the pound mass," and later adds: "When scientific accuracy is required we must correct for the change in the value of g due to any difference in altitude or latitude from those of the place to which the standard was originally referred." It is assumed by Thomson and Tait in the Natural Philosophy that: "In all cases where great accuracy is required the results obtained by such a method have to be reduced to what they would have been if the measurements of force had been made by means of a perfect spring-balance graduated so as to indicate the force of gravity on the standard weights in some conventional locality."

Let it rather be understood that, in the use of gravitation units, the pound when used as a unit of force is always local, and that in the formulæ the numerical value of m also implies a local unit of mass. But to avoid confusion it is best that in formulæ for application m should be replaced by W/g. And, in any comparison of results involving different values of g, let each force be expressed in poundals by simply multiplying by the local value of g.

Professor Greenhill has made a marked innovation in this matter as compared with the usage of recent English writers. Calling the pound the unit of weight, and refusing to regard weight as a force but rather as "the quantity which is determined by the operation of weighing," he practically makes the pound a unit of mass; and, abandoning the formula $W = mg$, puts W for the number of pounds, so that in formulæ it appears where we are accustomed to see m. Then, with regard to force he says "it is convenient to take the attraction of the earth on a pound weight as the unit of force, and to call it the *force of a pound;* this is the British unit of force in universal use in all practical problems of architecture, engineering, mechanics, and artillery."

As contrasted with the usual notation supposing absolute units to be employed, W is thus merely put in the place of m so that Wg instead of mg is the expression in poundals for the force of gravity acting on the body. At the same time, however, Professor Greenhill uses F, R, and other symbols of force as co-dimensional with W, so that they are the numbers of local pounds of force, and it must be remembered that the expressions for the same forces in poundals are Fg, Rg, etc. With regard to gravitation and absolute units, he says: "The attraction of the earth in any locality provides such a convenient and invariable measure of force that all instruments, great and small, for measuring force and work are calculated and graduated originally in gravitation measure; the reduction to absolute measure if required being made subsequently by means of the local value of g; presumably determined previously with the greatest attainable accuracy by means of pendulum experiments."

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, April 7, at half-past three o'clock, the president, Dr. McClintock, in the chair. Mr. Pomeroy Ladue, of the University of Michigan, having been duly nominated, and being recommended by the council, was elected to membership. The president announced the resignation on account of ill health of the treasurer, Mr.

Harold Jacoby, and stated that the council had appointed Professor R. S. Woodward as his successor, Mr. Jacoby taking the place thus left vacant in the council. A set of amendments to the constitution, recommended for adoption by the council, were read by the secretary.

Professor W. Woolsey Johnson read a paper entitled "Gravitation and absolute units of force." An abstract of this paper appears in the present number of the BULLETIN: see p. 197. In the discussion which followed the paper, Mr. C. S. Peirce proposed that the term *galileo* be applied to the unit of acceleration in the C. G. S. system.

Mr. C. S. Peirce exhibited an arithmetic of 1424, from the valuable collection of Mr. George A. Plimpton of New York. It is an extensive manuscript work written in Latin, and has been entirely unknown to the historians of mathematics. The author was Rollandus, a Portuguese physician, known for a work upon surgery and another upon physiognomy. He was a minor canon of the Sainte-Chapelle and a protégé of John of Lancaster, to whom the arithmetic bears a flowery dedication.

THE summer meeting of the NEW YORK MATHEMATICAL SOCIETY will be held this year on August 13, 14, 15, and not on August 20, 21, 22, as announced in the last number of the BULLETIN. The first general session of the American Association for the Advancement of Science will be held on August 16. At the time of making the announcement a month ago, it was anticipated that the American Association would meet one week later.

Dr. E. STUDY has been made Professor of Mathematics at the University of Bonn to succeed Professor Minkowski, who was called to Königsberg.

DURING the second semester of the current year the mathematical courses at the *Sorbonne,* as announced in *L'Indicateur des Cours publics,* are: Appell, Dynamics of systems. Boussinesq, Turbulent and whirling movement of fluids in conduits of great cross-section. Hermite, Theory of Eulerian integrals and Theory of elliptic functions. Picard, Differential equations from the point of view of mathematical physics. Poincaré, Calculus of probabilities. T. S. F.

THE German mathematicians will meet this year in Vienna, September 24–30. At this time will take place both the meeting of the *Deutsche Mathematiker-Vereinigung* and that of the first (mathematical) section of the *Versammlung deutscher Naturforscher und Aerzte.* Professor W. Dyck of

Munich (Hildegardstrasse 1½) is secretary of the former association. A preliminary programme of the proceedings will be issued early in July.

THE one hundredth anniversary of the birth of the Russian mathematician *Nicolas Ivànovich Lobachevsky*, which occurred on the 3d of November (or 22d of October, according to the Russian calendar) 1893, was celebrated in a worthy manner by the Physico-Mathematical Society of the University of Kazàn. The celebration extended over three days.

On Friday, the 3d of November, after religious services and a solemn requiem mass, the University Senate assembled to hold a memorial meeting at which Professors Th. M. Suvòrov and A. V. Vasiliev spoke on the life and scientific work of the Russian geometer. The daily paper "Volga Messenger" published a portrait of Lobachevsky on its front page and devoted over three pages to accounts of his varied activity as a man of science, as rector and professor of the university, and as a progressive and public-spirited citizen.

On the next day, the Physico-Mathematical Society arranged a special public session at which the following papers were read: (1) by *A. V. Vasiliev*, On Lobachevsky's writings on algebra and analysis; (2) by *I. A. Iznòskov*, On Lobachevsky's activity in the Imperial Economic (Agricultural) Society of Kazàn; (3) by *V. A. Sikstel*, The foundations of spherical geometry; (4) by *D. M. Sintsòv*, Lie's researches concerning the foundations of geometry; (5) by *Th. M. Suvòrov*, On differential linear elements.

In the afternoon of November 5th, the Municipal Council of the city of Kazàn dedicated with appropriate ceremonies a memorial tablet inserted in the front wall of the house in which Lobachevsky had lived. In the evening of the same day, at a public session of the Physico-Mathematical Society, the committee appointed to raise a "Lobachevsky fund" rendered its first account, and two more papers were read: (1) by *A. N. Smirnòv*, On the axioms of geometry, and (2) by *A. V. Vasiliev*, On n-dimensional geometry.

The contributions to the Lobachevsky fund had reached the sum of 3039.55 rubles at the time of the celebration; but the subscription is still open. About 200 rubles have been contributed by Americans. The income from this fund it is intended to use for the annual award of a Lobachevsky medal or prize, and, if possible, for erecting a bust of Lobachevsky at Kazàn, in the public square that bears his name.

Several years ago the Physico-Mathematical Society of Kazàn published a complete edition of Lobachevsky's geometrical works, in two volumes, the first volume containing the writings in the Russian language, the second those written in German or French (*Collection complète des œuvres géométriques de N. I. Lobatcheffsky*, édition de l'Université impériale de Kasan; 4to; Vol. I, 1883, viii and 550 pp., 14 plates; Vol. II, 1886, 8 pp., pp. 551–680, and xx pp., 4 plates, 1 portrait). Of the second volume, which contains a photographic portrait of Lobachevsky, a brief sketch of his life, and a bibliography of non-Euclidean geometry by M. E. Vashchenko-Zakharchenko, based on Halsted's, 50 copies are still for sale at the reduced price of 8 francs (including postage); the price for both volumes together has now been raised to 8.80 rubles. Orders for these works and contributions to the Lobachevsky fund can be sent to the Physico-Mathematical Society of Kazàn, Russia, of which Professor A. V. Vasiliev is the president. A. Z.

INSTEAD of the republication by the Leipzig Academy of Sciences of Lambert's paper described in the BULLETIN for December 1893, Dr. Paul Staeckel of the University of Halle and Professor Friedrich Engel of the University of Leipzig will publish within this year with Teubner of Leipzig a volume under the title *Die Theorie der Parallellinien*, containing Lambert's essay and the first book of the marvellous work by Saccheri, "Euclid vindicated from every fleck," wherein he developed in 1733 the two hypotheses which, besides Euclid's, are possible, and also obtained all the results in this subject which have been ascribed to Legendre.

The book will open with a collection of those theorems of Euclid which are related to his eleventh axiom (the parallel-postulate). In the time from Euclid to Wallis, expressions about the parallel-postulate are rare; but Wallis in 1663 showed that the axiom of the existence of similar figures is equivalent to it. After Wallis comes the wonderful Saccheri, 1733. Lambert in 1766 carried on Saccheri's ideas.

The relation between the different authors will be shown, and full notes, explanatory, historical, and bibliographic, will accompany the text.

LA Real Academia de Ciencias Exactas, Físicas y Naturales de Madrid offers first and second prizes and honorable mention for theses on the following subject:

Didactic treatment of the modern non-Euclidean geometric theories, or reasoned analysis of the principal works on this part of mathematical science.

The first prize consists in a special diploma, a gold medal of 60 grammes weight, and one thousand five hundred pesetas in money. It includes further the printing of the thesis at the cost of the Academy, and the delivery to the author of one hundred copies. The second prize is the same, except the money. Theses will be received until December 31, 1895.

G. B. H.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Amodeo (F.). Le operazioni sui numeri intieri. Napoli, 1894. 8vo. 50 pp. Mk. 1.50

Caronnet (T.). Recherches sur les surfaces isothermiques et les surfaces dont les rayons de courbure sont fonctions l'un de l'autre. [Thèse.] Paris, Gauthier-Villars, 1894. 4to. 73 pp. Illustrated.

Descartes (R.). Die Geometrie. Deutsch herausgegeben von L. Schlesinger. Berlin, Mayer und Müller, 1894. 8vo. 11 and 116 pp. 2 plates. Mk. 3.60

Grotendorst (N. C.). Beginselen der Differentiaal- en Integraal-rekening. Uitgegeven voor rekening der Koninklijke Militaire-Academie. Breda, 1894. 8vo. Mk. 9.50

Heffter (L.). Einleitung in die Theorie der linearen Differentialgleichungen mit einer unabhängigen Variablen. Leipzig, Teubner, 1894. 8vo. 14 and 258 pp. Mk. 6.00

Hime (H. W. L.). The outlines of quaternions. London, Longmans, 1894. 8vo. Cloth. 10s.

Katalog der auf Hamburger Bibliotheken vorhandenen Litteratur der reinen und angewandten Mathematik und Physik. Herausgegeben von der Mathematischen Gesellschaft in Hamburg anlässlich ihres 200-jährigen Jubelfestes. Nachtrag I. Hamburg, 1894. 8vo. 88 pp. Cloth. Mk. 2.00

Kronecker (L.). Vorlesungen über Mathematik. Herausgegeben unter Mitwirkung der Königlichen Akademie der Wissenschaften zu Berlin. Vol. I: Vorlesungen über die Theorie der einfachen und der vielfachen Integrale. Herausgegeben von E. Netto. Leipzig, Teubner, 1894. 8vo. 10 and 345 pp. Mk. 12.00

Peano (G.). Notations de logique mathématique: introduction au formulaire de mathématique, publié par la *Rivista di Matematica.* Turin, Guadagnini, 1894. 8vo. 52 pp. Fr. 3.50

Riccardi (P.). Biblioteca Matematica Italiana dalla origine della stampa ai primi anni del secolo XIX. Ripubblicata à cura della Società Tipografica Modenese, con 2 nuove serie di aggiunte. 2 parti. Torino, 1894. 4to. Mk. 42.00

Van Vleck (E. B.). Zur Kettenbruchentwickelung Lamé'scher und ähnlicher Integrale. [Dissertation, Göttingen.] Baltimore, Friedenwald, 1893. 4to. 92 pp.

Vivanti (G.). Il concetto d'infinitesimo e la sua applicazione alla matematica; saggio storico. Mantova, 1894. 8vo. 134 pp. Mk. 3.00

Vyvyan (T. G.) Analytical geometry for beginners. Part I: The straight line and circle. London, Bell [New York, Macmillan], 1894. 12mo. 130 pp. (Cambridge mathematical series.) Cloth. $0.60

Wettum (T. B. van). Researches on matrices and quaternions. Leyden, 1894. Folio. 4 and 27 pp. 1 plate. Mk. 2.00

II. ELEMENTARY MATHEMATICS.

Angiolini (G.). La trigonometria secondo i programmi per il liceo, esposta con metodo chiaro e breve. Napoli, Cosmi, 1893. 16mo. 22 pp. 1 plate. Fr. 0.75

Borth (E. F.). Die geometrischen Constructionsaufgaben, methodisch geordnet und mit einer Anleitung zum Auflösen derselben versehen. 8te Auflage. Leipzig, 1894. 8vo. 8 and 167 pp. Illustrated. Cloth. Mk. 1.80

Bradbury (W. J.) and **Emery** (G. C.). Algebra for beginners. Boston, Thompson, Brown & Co., 1894. 12mo. 150 pp. Cloth. $0.60

Bussler (F.). Sammlung mathematischer Aufgaben. Für den Gebrauch in den oberen Klassen höherer Lehranstalten zusammengestellt. Dresden, Ehlermann, 1894. 8vo. 4 and 120 pp. Mk. 1.40

Collins (J. V.). Text-book of algebra. Through quadratic equations. Chicago, Albert, Scott & Co., 1893. 8vo. 463 pp.

Del Vecchio (P.). Elementi di aritmetica ragionata, applicati alla pratica. Lucera, 1894. 8vo. 89 pp. Mk. 2.50

Diesener (H.). Die ebene Geometrie. Praktisches Unterrichtsbuch zur leichten Erlernung der Planimetrie. Bearbeitet für den Selbstunterricht und zum Gebrauche an Baugewerkschulen und Fortbildungsschulen. 2te Auflage. Halle, Hofstetter, 1894. 8vo. 3 and 138 pp. Illustrated. Mk. 2.70

Franchetti (G.). Cenni storici sulle matematiche elementari. Sassari, Satta, 1893. 8vo. 68 pp. Fr. 5.00

——— Delle curve generate da un punto del piano di un circolo quando questo ruota su linee date nel suo piano. Sassari, Satta, 1893. 8vo. 55 pp. 1 plate. Fr. 5.00

Gomez y Pallete (J.). Trigonometria rectilinea y esférica, considerada como introducción al estudio de la geometria analítica. 8a edición. Madrid, 1894. 4to. 3 and 128 pp. Illustrated. Mk. 6.00

Hayes (E.). Lessons on higher algebra, with an appendix on the nature of mathematical reasoning. Revised edition. [Boston,] Norwood Press, 1894. 8vo. 12 and 96 pp.

Heilermann (H.) und **Diekmann** (J.) Lehr- und Uebungsbuch für den Unterricht in der Algebra an den höheren Schulen. Teil II: Die Gleichungen zweiten Grades mit mehreren Unbekannten; die Progressionen; die kubischen und biquadratischen Gleichungen; niedere Analyse. 4te Auflage. Essen, Baedeker, 1894. 8vo. 5 and 188 pp. Mk. 2.25

Martus (H.). Leitfaden für den Unterricht in der Raumlehre. 2ter Teil: Dreiecksrechnung und Körperlehre. Bielefeld, Velhagen und Klasing, 1894. 8vo. 5 and 138 pp. Mk. 1.80

Mehler (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Berlin, Reimer, 1894. 8vo. 8 and 266 pp. Illustrated. Mk. 1.60

Pfeiffer (E.). Lehrbuch der Arithmetik und Algebra für höhere Schulen, nach methodischen Grundsätzen bearbeitet und herausgegeben. 1ter Teil, 2te Auflage. Jena, Mauke, 1894. 8vo. 8 and 145 pp. 4 plates. Mk. 2.40

Reum (A.). Der mathematische Lernstoff für den Unter-Secundaner des Gymnasiums, des Realgymnasiums und der Oberrealschule, sowie für den Primaner der Realschule. Essen, Baedeker, 1894. 8vo. 7 and 44 pp. Illustrated. Mk. 0.60

Schwering (K.). Anfangsgründe der analytischen Geometrie für höhere Lehranstalten. Nach den neuen Lehrplänen bearbeitet. Freiburg in Baden, Herder, 1894. 8vo. 7 and 24 pp. Illustrated. Mk. 0.40

——— Stereometrie für höhere Lehranstalten. Nach den neuen Lehrplänen bearbeitet. Freiburg in Baden, Herder, 1894. 8vo. 51 pp. Illustrated. Mk. 0.80.

Weinmeister (P.). Sammlung mathematischer Formeln und Sätze. Ein Nachschlage- und Wiederholungsbuch für Schüler höherer Lehranstalten. Leipzig, Siegismund und Volkening, 1894. 8vo. 38 pp. Mk. 1.50

Weissenborn (H.). Die Berechnung des Kreisumfanges bei Archimedes und Leonardi Pisano. Berlin, 1894. 8vo. 32 pp. Mk. 1.50

Wentworth (G. A.) and **Hill** (G. A.). An examination manual in plane geometry. Boston, Ginn, 1894. 8vo. 3 and 138 pp. Cloth. $0.55

Zwicky (M.). Grundriss der Planimetrie und Stereometrie nebst Uebungsaufgaben. 2ter Teil: Stereometrie. Bern, Schmid, Francke und Co., 1894. 8vo. 5 and 64 pp. Illustrated. Mk. 1.20

III. APPLIED MATHEMATICS.

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THE LUNAR THEORY.

Traité de Mécanique Céleste. Vol. III. Perturbations des planètes d'après la méthode de Hansen; théorie de la lune. By F. TISSERAND. Paris, Gauthier-Villars, 1894. 4to, pp. 424.

IT is somewhat strange that a subject like the lunar theory, which has received so much attention since its first principles were given by Newton, should be allowed to pass its second centenary before the appearance of a treatise like the present one. Every writer on the subject up to the present time has given a theory from one point of view only—either his own or further developments of some previous one. It has been reserved for M. Tisserand to give us in a collected form the principal methods which have been used to obtain a practical solution of a problem which can be stated in the very simplest terms, but which is perhaps one of the most difficult to master in the whole range of applied mathematics. And its difficulties are not merely those of simple analysis. The results obtained have been generally considered of little value unless they either assist us in the solution or are of value in the practical applications or finally help towards giving a geometrical (intuitive) idea of the meaning of the various expressions arrived at. The geometrical representations have always been kept prominently in the foreground, perhaps owing to Newton's influence, whose idea of the variable ellipse, developing itself naturally from his methods, has been used almost without exception by every writer since his time. In the last few years some doubts have been raised as to whether the ellipse is the most valuable starting-point, and of these I shall speak later in connection with the works of Adams, Gyldén, and Hill. At the same time it must not be forgotten that the complete problem of the motion of the moon is a single one only in the sense that the erection of a house is so—we might use perhaps one kind of material only which may suffice for every portion, but which would hardly produce the most finished results. And thus M. Tisserand in making a collection of methods is not merely giving us a series of disconnected solutions, but furnishes the materials for a building which completed would comprise parts of perhaps every one. This is well illustrated by Dr. Hill's contributions in recent times. He uses in giving a new method of initial treatment Euler's plan of dividing up the various classes of inequalities: for the class of inequalities depending on the figure of the earth Delaunay's method of the variation of arbitrary constants, and, for the inequalities produced by the motion of the ecliptic, Laplace's method. In each of these cases there is a special advantage depending on the part of the problem to be solved.

From the time of Newton to the beginning of the present century it is to the French that we owe most of the advance made in the development of the theory. The magnificent genius of Newton in presenting all his discoveries in a geometrical form and absolutely covering up all the traces of the analysis by which he reached them seems to have raised a barrier against advance in England. His successors, unable to proceed on geometrical lines and gifted with obstinacy in keeping to his methods without recognizing what underlay them, made no progress. In fact the method of Newton which was constructed in order that he might make himself understood and also avoid controversy was a somewhat costly one for England. It fulfilled its design, but the cure became a new disease, and as such remained until Babbage and Peacock brought in the calculus methods, long familiar to their Continental contemporaries.

The history of the lunar theory after Newton gave the "Principia" to the world in 1684–6 embraces several periods. It was almost untouched for over half a century until Clairaut, d'Alembert, and Euler, within a few years of one another, published theories on the motion of the moon. In these we have practically the first attempts to apply analysis in the modern sense and to determine definitely the principal inequalities from elliptic motion. This constitutes the second stage. In the third we must place first Laplace, who extended the theory in every direction, taking into account such inequalities as might be produced by the action of the planets, and in particular assigning a reason for the secular acceleration of the moon's mean motion. It is well known now how theory explains only half this acceleration, and the curious accident of the first approximation apparently accounting for the whole of it is one of those points in scientific history of which instances are not wanting elsewhere. His successors Poisson, Plana, de Pontécoulant, Lubbock, and Damoiseau aided in carrying the theory further, and of these perhaps the most notable was de Pontécoulant, who gave a complete analytical treatment of the motion as affected by the sun.

To the fourth period must belong Hansen, Delaunay, and many other names of lesser note. The two mentioned gave entirely new methods of treatment, each of which stands to this day unrivalled for the particular object pursued. Hansen, though he worked out his theory only numerically, did so with a completeness and accuracy which has satisfied most of the needs even of the present time. On the other hand, Delaunay conceived the idea of a complete analytical development, the immense labor of which occupied him twenty years, and as a single monument of patient labor it stands without parallel. Though some of the coefficients are not perfectly satisfactory, their general agreement with Hansen's is so good that most of them may satisfy our requirements for some time to come.

The fifth and last period was opened in 1877 by the publication by Dr. Hill of his two papers, "Researches in the lunar theory" and "The part of the motion of the Lunar Perigee which is a function of the mean motions of the Sun and Moon." A new lease of life was given to the whole subject when these appeared. Adams, who had a great aversion to publishing any results except such as were in a very complete state, gave the value which he had obtained for the similar part of the motion of the node, working on somewhat the same lines. Since that time most of the investigations, chiefly by Poincaré, have turned on the convergence of the series employed and the legitimacy of the methods used.

M. Tisserand's plan may be briefly sketched. The main portion of the book consists of accounts of the various methods adopted to solve the problem of three bodies only as far as it refers to the sun, earth, and moon. Under these different methods are given, as appears most convenient, those portions of the perturbations of the moon which proceed from other causes, as the action of the planets, the influence of the figures of the earth and moon. The natural result of such a treatment is a certain want of continuity which, to a student reading the subject for the first time, would be bewildering. But for such a reader the book cannot be intended. Not one of the least difficulties to be encountered is the meaning of the various constants in which the results of a given theory are finally expressed and their connection with those of other methods, and this difficulty is hardly touched upon at all. Again, questions of purely mathematical interest are entirely omitted. It is not intended by these remarks to attempt to throw any discredit on M. Tisserand's labors, but rather to show that there is ample scope for a treatise on different lines more adapted to the needs of a student.

Chapters I and II are devoted to a short study of the equation

$$\frac{d^2x}{dt^2} + x(q^2 + 2q_1 \cos 2t + 2q_2 \cos 4t + \dots) = 0,$$

$q, q_1, q_2, \dots$ being constants. The first chapter is not altogether clear. It is complicated by the fact that M. Tisserand deals with the equation in three cases, namely, when $q_2 = q_3 = \dots = 0$, when the number of quantities q is finite, and lastly when their number is infinite. And in some of the discussions it is not easy to see whether the conclusions for one case apply to the others, or indeed which case is for the moment being treated. The equation is integrated and expansions are given for the results. The second chapter is devoted to the case when the number of quantities q_i is infinite, and in it the expansion of the infinite determinant is explained with admirable clearness.

In chapter III we are given a few details of Newton's method of treating the lunar theory. Here M. Tisserand, in setting forth a general idea of the method, seems to have preferred to follow the analytical translations, chiefly using that of Laplace. The effect is perhaps to do Newton an injustice. It is not made clear that Newton in the first instance worked out his results by some kind of analytical method, and that the feat of the "Principia" was the geometrical portion. We lose thus much of its pleasant antique flavor. And though we are given one or two examples of Newton's geometry it seems hardly necessary to develop the results analytically, at any rate, not in this chapter. Newton's objections to analysis were rather from the point of view of mental development than from any lack of appreciation of its value as a method of research. He valued highly the habit of mind and intellectual grasp obtained by a "picture view" of any problem. In this perhaps in later times we have been somewhat negligent, though Riemann and others have done much to bring it to the front again. Perhaps a selected portion of the "Principia" would still be the most valuable way of initiation into the lunar theory.

The theories of Clairaut and d'Alembert occupy chapter III, the greater portion of which is devoted to that of the former. Clairaut was the first to publish a method founded on the integration of the differential equations of motion. He employed as independent variable the true longitude of the moon, and he first showed that the observed motion of the perigee could be fully accounted for by the law of Newton only. D'Alembert's method is similar in principle. He observed that we can express all the perturbations by means of sines and cosines of four arguments. M. Tisserand shows us how doubts still existed as to the sufficiency of Newton's law in accounting for all the principal perturbations. Clairaut had in fact found the effect of introducing a term depending on the inverse cube when it occurred to him to go to a further approximation with the Newtonian law only, and the apparent discordance between theory and observation in the motion of the perigee was soon accounted for.

In chapters V and VI we have the two theories of Euler published in 1753 and 1772, the former with an appendix in which he outlined other methods of treating the subject. It is this appendix which M. Tisserand uses, and in it the method of the variation of arbitrary constants makes its first appearance. In the second theory we find rectangular coördinates in use with the mean anomaly of the sun as independent variable.

The next chapter contains an exposition of Laplace's method of proceeding as given in the classic "Mécanique Céleste." Though the method of using the true longitude as independent variable simplifies the equations of motion and gives developments which are somewhat more convergent than those in terms of the time, it possesses a fatal defect due to the fact that a reversion of series is necessary in order to express the coefficients directly in terms of the time. To the extent of a small number of approximations this defect is not very evident. The only text-book in English, Godfray's "Lunar Theory," has adopted this method.* There can be little doubt that, as far as the solar perturbations are concerned, it is practically useless where great accuracy is required. In this chapter Laplace's discovery of the secular acceleration is given a slight treatment in preparation for a full explanation and development later.

With Laplace's theory we are given those of Damoiseau and Plana, the former of whom used Laplace's method with numerical values, the latter giving algebraic results and substituting numbers only at the conclusion. A table is given of one hundred and sixteen of Hansen's coefficients in longitude, with columns denoting the differences between his results and those of Damoiseau's theory, Damoiseau's tables (1828), de Pontécoulant, Plana, and Delaunay. Prof. Newcomb's transformation of Hansen published in vol. I of the "Astronomical Papers" is used.

Then follows an interesting chapter on a variety of topics which have been treated more recently by Laplace's method. We have Gyldén's "intermediary" orbit where the equation solved in chapter I arises. Such an orbit is taken as fundamental in the same sense as the elliptic one is in most of the other theories. M. Tisserand rightly thinks that the introduction of elliptic functions is unnecessary, and he touches a question in this matter that is of vital importance as far as the best mode of expression in the lunar theory is concerned. Take, for instance, two different intermediary orbits, the first an ellipse and the second Dr. Hill's periodic curve, depending on the ratio of the mean motions (m); the coefficients of these are functions of e and m respectively. Either is well and easily expressed, the first by means of elliptic or Besselian functions and the second by sines and cosines with coefficients which are series with a fair degree of convergence. Yet if we try to combine these so as to express the terms depending on *both* e and m by means of these functions we meet with many difficulties. To express the results as a function of these two curves gives an analytical expression which would be of little use owing to its complicated quality, and we fall back on the old method of using one or the other as a primary orbit. The question narrows itself then down to the best choice of a primary or "intermediary" orbit. These orbits have been given some attention by Gyldén and Hill.

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* Owing to our method of measuring time we have become so accustomed to regard it as a fundamental conception which is not capable of being replaced by any other that there exists a real mental difficulty in considering the time as a function of the true longitude—a difficulty which scarcely appears when we consider the true longitude as a function of the time. And in working out the perturbations of the time when expressed in the former manner it is not easy to recognize their meaning. For this reason Godfray's text-book lies under a great disadvantage from a beginner's point of view. The analytical developments are not difficult, but until the reversion of series has been made it is not easy to understand the geometrical meaning of the various inequalities. For teaching purposes a theory like that of de Pontécoulant would seem to be much more advantageous.

To illustrate Poisson's method, that of the variation of arbitrary constants, M. Tisserand finds the inequalities due to the shape of the earth, the secular motion of the node and perigee due to the action of the sun, the influence of the displacement of the ecliptic, and a few long-period inequalities. They are worked out in one or two cases with sufficient practical accuracy. Then follows a chapter on Lubbock and de Pontécoulant which gives the method of the latter. We are shown the complete first approximation into which no serious difficulties enter. Any trouble that the method entails enters in the higher approximations. It seems as if we might have had one or two classes of inequalities only worked out for a first approximation and the space thus saved devoted to a second approximation of one class, say that depending on m and e or m and γ.

In chapters XI and XII Delaunay's theory is very fully developed. M. Tisserand uses for this purpose the "principal function," and by its means avoids some of the tedious transformations of Delaunay. These can be also managed easily by the use of the calculus of variations, and perhaps the results of Art. 94 might have been obtained more briefly in this way. It may be mentioned that Hayward* obtained the primary set of equations immediately from geometry.

An extremely interesting chapter follows devoted to the secular acceleration of the moon's mean motion. In giving a slight sketch of its discovery by observation M. Tisserand remarks how narrowly Lagrange missed making the discovery which was reserved for Laplace. On the ancient eclipses some remarks are made which put their value in such a clear light that I believe it worth while to reproduce them. "If," he says, "we pay attention to the known fact that in every eclipse of the sun the regions of the earth covered by the pure shadow of the moon (regions of totality) form a very narrow band; that in a given place total eclipses of the sun are very rare (at Paris, for instance, one only has been visible during the eighteenth and nineteenth centuries); it will be understood that if we know positively that an eclipse of the sun has been observed at a given place at a period not exactly fixed but nevertheless comprised between limits not too large, we can deduce exact conclusions about the values of certain elements of the lunar orbit at this period, and in particular about the corresponding mean motion. With a small variation in this latter quantity, the given eclipse *calculated by the tables* might not have been total at this place. At the same time there is room for doubt in other directions, namely, the question of the unknown date, the want of definiteness in giving the place of observation, the vagueness of the record, and the fables which always tend to enter in these cases, so that we cannot always be certain if there really was a total eclipse. Also the errors of the tables play an important part, and we can understand from all these considerations that the method gives in reality less than it seems to promise."

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* *Quarterly Journal of Mathematics*, vol. 3, p. 22.

In relating the discovery of Adams as to the inexactness of the previous theoretical values, M. Tisserand relates graphically how Plana, who was most directly concerned, first recognized his error and then found a new value different from his own old one and also from that of Adams; how Delaunay, Lubbock, Cayley, Hansen successively proved that Adams was right, and how since then the theoretical value has been calculated with all the exactness which we can ask for. He then gives Delaunay's calculations for finding the term in m^4 by Poisson's method, and finally, what is perhaps as useful as all that precedes, tells what attempts have been made to account for the still unexplained portion.

In chapter XIV we are given in full the periodic solution as found by Hill which depends on the mean motions of the sun and moon only. No reference is made to the portion of his work on moons of other lunations which perhaps have a theoretical rather than a practical interest. (The numerical values found are given correctly with one exception which might cause a misapprehension. On page 270, equation (34), in the value of $r/a_0 \sin (L - nt - \epsilon)$, the constant term unity should be omitted.) Then follows a chapter containing the investigations of the same writer on the Perigee of the Moon. The celebrated equation

$$\frac{d^2 W}{d\tau^2} + \Theta W = 0$$

has been given a definite geometrical meaning by Adams, in his lectures delivered in 1885 and since independently by Prof. G. H. Darwin in his lectures of this year. It will probably be published with the collection of papers left by Adams now in course of preparation for the press. Adams obtains the equation in the form

$$\frac{d^2 \omega}{dt^2} + P\omega = 0,$$

where ω is the displacement from the primary orbit in a direction along the normal to that orbit. When ω has been found, the displacement along the tangent is obtained by a simple integration.

The work of Adams on the motion of the node and his two theorems on the constant part of the inverse of the radius vector occupy another chapter, while a further chapter is devoted to an explanation, or rather a development, of Hansen's theory which is not easy to follow. The large number of symbols used necessarily complicates it, and matters are not improved by the recurrence in two or three cases of the same symbol with different meanings. The theory is, however, given very fully, and concludes with Prof. Newcomb's transformations for comparing Delaunay and Hansen as given in the "Astronomical Papers."*

Chapter XVIII treats of the planetary inequalities in the motion of the moon, and is developed mostly according to M. Radau's method. The principal long-period inequalities are given and discussed with some detail.

The last chapter on the actual state of the theory of the moon will be found the most interesting of any in the volume. A large portion of it is devoted to the secular acceleration as shown by ancient eclipses, and naturally is chiefly based on Prof. Newcomb's great investigation. There are also in this chapter, as in other parts of the volume, some of M. Tisserand's own contributions, and a single reading will show that it is written by one who has not merely made a compilation of the works of others, but who possesses in addition that intimate acquaintance with the subject which is only obtained by those who have advanced it at least in some directions. Whatever the defects of the book may be, it will take a high rank amongst the many classic treatises on celestial mechanics. If we might venture to make a suggestion, it would be that a full alphabetical index of subjects and names be appended to the fourth volume, thus adding greatly to the value of the series as books of reference. The increasing labor of finding out what has been done in a subject makes such an index almost necessary.

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* It would be difficult to find four consecutive volumes of any publication concerned with a single science with more valuable results than those published at Washington under the title of "Astronomical Papers prepared for the use of American Ephemeris and Nautical Almanac."

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BACHMANN'S THEORY OF NUMBERS.

Die Elemente der Zahlentheorie. Von PAUL BACHMANN. Leipzig, Teubner, 1892. 8vo, xii + 264 pp.

"*The most beautiful theorems of higher arithmetic have this peculiarity, that they are easily discovered by induction, while on the other hand their demonstrations lie in exceeding obscurity, and can be ferreted out only by very searching investigations. It is precisely this which gives to higher arithmetic that magic charm which has made it the favorite science of the first mathematicians, not to mention its inexhaustible richness, wherein it so far excels all other parts of pure mathematics.*"—GAUSS.

Interest in the theory of numbers has, perhaps naturally enough, not always remained at the high-water mark indicated in the above encomium, though there is not wanting a list of illustrious names, reaching down to the present generation, of those who have made important contributions to the theory. Within the last decade there have appeared translations of classic works of the highest importance, one at least grown well-nigh inaccessible in the original, as well as quite a number of text-books on the subject. During the last three years there have appeared text-books, each with its own points of merit, in the languages English, French, and German.* All these recent publications point to a strong interest in the theory of numbers now existent, and at the same time give promise that the increased facilities for study will give a fresh impetus to research in this field.

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* Respectively by Mathews, Lucas, and Bachmann.

We consider in this review one of these recent text-books on the theory of numbers. This work is the beginning of a homogeneous presentation of the subject in its present status, the object being rather to make a synopsis or an outline of this branch of mathematics than a compendium containing all that has ever been written on it. The volume before us treats of the *elements* of the subject—congruences, quadratic residues, and forms, the last being restricted to *binary* forms. Consequently the division of classes into *genera* has been excluded, because the fundamental theorem proving the actual existence of the possible genera is based upon the consideration of ternary forms. Of course, all non-arithmetical proofs are also strictly excluded.

Regarding the work as a whole, the following points deserve especial remark:

I. *Clearness of Presentation.*—This is characteristic not only of the discussions, but also and especially of the formulations of the problems and the analyses of methods. This, together with the well-balanced mean preserved between excessive conciseness and too great diffuseness, makes the book unusually readable. It must be said, however, that the simpler presentation is not always preferred, but rather that which fits more consistently into the general plan of treatment. Instances of this will be noticed in the detailed sketch to follow.

II. *Introduction of Recent Results.*—Since the time of Gauss not much has been accomplished, relatively speaking, by elementary methods in the elementary field as outlined above, but still the author finds sufficient opportunity for presenting recent results to warrant the statement that he has given us, as proposed, not simply a sketch of the subject as Gauss left it, but of the elementary theory of numbers in its principal outlines as it stands to-day. Particular mention may be made of the application of the theory of groups, and the generalization of the Gaussian lemma by Schering and Kronecker. Throughout the book well-chosen references direct the reader to original sources.

III. *Historical Remarks.*—Concise sketches of the salient features of the historical development of the subjects considered are made as occasion arises and add materially to the interest and also to the clearness of the treatment. The author is usually scrupulous in crediting even simple and commonly current results to their original publisher (e.g., pp. 19, 25, 26). On this account only, it is worth mentioning that the proposition (p. 30) concerning the highest power of a prime contained in $m!$ was given by Legendre.*

We proceed now to a more detailed consideration of the contents of the book, dwelling only upon points whose treatment differs from that usually found in other books.

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* Théorie des Nombres, Introd. xvi.

In an introductory section the concept "number" is briefly taken up (pp. 1–4). It has been customary with writers of treatises on the theory of numbers tacitly to assume the system of positive integral numbers. It is questionable whether so abstract an idea as "number" can be adequately treated in four pages, and whether our author has really done more than to occupy himself throughout four pages in plausibly assuming the system of positive integral numbers. The operations, addition and multiplication, with their laws are also discussed in the introduction.

SECTION I. *The Divisibility of Numbers.* — The author begins by considering the least positive remainders which the entire system of positive and negative integers leave when divided by a given integer n, the latter being at once called the *modulus.* This brings into prominence the periodicity of the series of integers taken modulo n, and various elementary properties of numbers with regard to divisibility are deduced by considering the least positive residues (mod n) written at the corners of a regular n-gon inscribed in a circle, a well-known method which the author credits to Poinsot.[*] The resolution of numbers into prime factors is shown to be unique, the number of divisors of a given integer and their sum are found, and properties of the greatest common divisor and of the least common multiple of given numbers are discussed. The highest power of p, a prime, contained in $m!$ is next found, and the result is applied in a purely arithmetical proof that the following expressions are integers:

$$\frac{m!}{r!\,s!\ldots t!}, \qquad (m = r + s + \ldots + t,)$$

[†] $$\frac{2a!\,2b!}{a!\,b!(a+b)!}.$$

The section closes with the evaluation of the expression $\phi(n)$ by a method different from that ordinarily used.

SECTION II. *Congruences.* — In establishing the *preliminary* theorems concerning congruences stress is laid upon the fact that we have really to do, not with the numbers themselves, but simply with the classes to which they belong (mod m). The fact that the classes which are characterized by residues prime to the modulus have a definite law of multiplication which is associative and "invertible,"* together with the fact that their number finite assumes that they form a finite *group*. The notion of a group is accordingly introduced, and some of the elementary properties of groups are set forth. The results are applied in achieving the solution of linear congruences. Of Fermat's theorem and its generalization three different proofs are given and analyzed. Kronecker's theorem† concerning the generation of groups whose elements are commutative with each other follows with somewhat more detail than in its original presentation. This amplification of this important theorem is welcome, though no amplification can remove the inherent abstractness of the subject. It would perhaps have been of advantage to bring forward explicitly the idea of a quotient-group recently introduced by Hölder,‡ which is fundamental for the whole proof. The theorem is at once applied in the proof of the existence of primitive roots for a prime modulus, and later in the consideration of the same question for a modulus which is a power of a prime. The existence of primitive roots for any modulus is discussed, and the properties and applications of indices are briefly presented.

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[*] *Liouville's Journal,* vol. 10.

[†] Catalan, *Nouv. Ann.* 1874.

SECTION III. *Quadratic Residues.*—Quadratic residues are defined, Euler's criterion for the possibility of the congruence $x^2 \equiv n \pmod{p}$ is set up, Legendre's symbol is introduced, the congruence $x^2 \equiv n \pmod{m}$ is considered for composite moduli of various forms, and then the problem is attacked: "To determine of what moduli a given number n is residue (or non-residue)" Restricting consideration first to prime moduli, we ask successively, "Of what odd primes is the numbers, $-1, 2$, any other prime q, residue (or non-residue)?" For the first two cases the question is satisfactorily and readily answered, while the third leads to Legendre's *law of reciprocity* of quadratic residues. The consideration of this important theorem is begun by proving the "Gaussian lemma," that $\left(\frac{n}{p}\right) = (-1)^\mu$, where μ is the number of negative numbers in the series of absolutely least residues (mod p) of

$$n,\ 2n,\ 3n,\ \ldots\ \frac{p-1}{2}\cdot n.$$

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* I have ventured so to express "eindeutig umkehrbar," ...

Before proceeding to a proof of the law of reciprocity, the author classifies the proofs which have been given into four categories:

First Category.—The first proof of Gauss, simplified by Dirichlet, constitutes a category by itself. Of all the proofs ever given, it attacks the problem most directly, and by induction achieves the result through simple conclusions from the idea of a quadratic residue.

Second Category.—The proofs of this category are based upon the theory of quadratic forms. To this category belong the second proof of Gauss, those of Kummer, and that of Legendre (incomplete).

Third Category.—Proofs directly connected with the division of the circle. They are exemplified by the fourth and the sixth proof of Gauss.

Fourth Category.—Proofs based on the Gaussian lemma. Examples, the third and the fifth proof of Gauss.

As the first category is treated in full in Dedekind-Dirichlet's text-book, and as the second and the third category are not elementary, they are (with one exception in the second category to be noticed later) excluded from detailed consideration.

Of the fourth category, the proof of the pastor Zeller is given.[*] This proof, though simple and elegant, has not before, as far as the reviewer knows, been presented in a text-book on the theory of numbers.[†] There follow Jacobi's generalization of Legendre's symbol and the generalized law of reciprocity, Eisenstein's algorithm for the determination by means of the law of reciprocity of the value of $\left(\frac{m}{n}\right)$, m and n being any two relatively prime integers of which at least one is positive, while with Schering's and Kronecker's generalization of the Gaussian lemma and, based on it, Schering's proof of the generalized law of reciprocity, and Kronecker's representation of the symbol $\left(\frac{Q}{P}\right)$ (Q and P being any two positive, odd, relatively prime integers) as the sign of certain products and the consequent simplification of Gauss' third proof, and as an application of Kronecker's representation another proof by Kronecker of the law of reciprocity, bring us into the field of the most recent researches in this subject.

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[*] *Berl. Montasber.* 1872.

[†] It may be of interest to give in this connection a list of the proofs presented in various text-books:

Tschebyscheff, Gauss III.
Dedekind-Dirichlet, Gauss I, III, IV.
Serret (*Alg. Sup.*), Gauss III.
Bachmann (*Kreistheilung*), Gauss VI, Eisenstein (*Crelle* 27). Liouville, Eisenstein (*Crelle* 29). Gauss IV (sketch).
Wertheim, Gauss V, Eisenstein (*Crelle* 28).
Mathews, Gauss I, III, Eisenstein (*Crelle* 27, in part).

SECTION IV. *Quadratic Forms.*—Varying the usual order, the author takes up first the representation of numbers by means of quadratic forms, and from this passes to transformation and equivalence of forms. Connection is made with quadratic residues by noticing that the question whether or not the *congruence* $x^2 \equiv n \pmod{m}$ can be satisfied is equivalent to the question whether or not the *form* $x^2 - n$ can represent at least one number which is a multiple of m. After defining forms, primitive and derived, properly and improperly primitive, the subsequent considerations are confined to *properly* primitive forms, i.e., forms

$$ax^2 + 2bxy + cy^2,$$

in which a, $2b$, c have no common divisor other than unity. The various cases of the value of the determinant $D = b^2 - ac$ are discussed, and, omitting less important theorems, the following fundamental results are reached:

I. *That the number m be properly representable by (a, b, c) of determinant D, it is necessary, though not sufficient, that D be quadratic residue of m.*

II. *Every proper representation of m by (a, b, c) belongs to a definite root of the congruence*

$$1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad x^2 \equiv D \pmod{m}.$$

This affords basis for the classification* of all possible representations of m by the form (a, b, c).

III. *All representations belonging to the same root of the above congruence are deduced from any one of them by means of the solutions of*

$$t^2 - Du^2 = 1.$$

This is the well-known Pellian equation, and its solutions are next found. In the only difficult case, viz., $D > 0$, it is shown, according to Dirichlet, how to derive all solutions from the fundamental solution; i.e., that in which u has its smallest value.

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* The author says "into groups." This is not happy, because the technical meaning of the term group has been used, and it is not shown that the classes of representations are, technically, a group, no multiplication law being defined.

In Nos. 12 and 14 of this section the method is indicated of finding all the representations of m by (a, b, c), $D > 0$, and then we pass to the " equivalence of quadratic forms," defined as follows:

Two quadratic forms of the same determinant are equivalent if every number which can be represented by the one can be represented also by the other by a representation belonging to the same root of the congruence 1) *as the first representation.*

With this definition as a basis, various particular theorems are established and illustrated, and finally the general result is reached:

Two forms of the same determinant are equivalent as soon as there exists at least one number m which can be represented by each form by a representation belonging in both cases to the same root of the congruence 1).

The subject of equivalence is concluded by translating the above *arithmetical* condition of equivalence into the ordinary *algebraic* condition, viz.:

The necessary and sufficient condition for the equivalence of two forms is that one can be transformed into the other by the substitution $x = \alpha x_1 + \beta y_1$, $y = \gamma x_1 + \delta y_1$, *where* $\alpha\delta - \beta\gamma = 1$.

The idea of equivalence affords a basis for the division of the infinity of properly primitive forms of the same determinant into *classes;* the forms of each class being equivalent with one another and with no others. The proof is next given that for every determinant *D, the number of classes is finite.* The next main problem considered is the determination of the representations of m by a *system of forms* consisting of one and only one form from each class of determinant *D*. The method is illustrated at length in an example. The decomposition of numbers into the sum of two squares is next discussed, improper representations being also considered in this case. After ambiguous forms and classes have been defined and illustrated, the problem of *composition of forms* is taken up. The method of composition having been defined, it is shown that the *class* of the resulting form is dependent only on the *classes* of the forms composed, that the process of composition is associative, commutative, and invertible. On account of these properties it is preferred to call the process *multiplication* rather than (with Gauss) *addition* of forms. From these results it follows that *the totality of equivalent properly primitive forms of given determinant constitutes a group,* and indeed a group whose elements are all commutative with one another. We see then immediately from Kronecker's theorem on commutative groups that certain classes can be selected such that all the others can be derived by composition of the selected classes with themselves. In a paper on the composition of forms,* Schering showed how "to set up a complete system of fundamental classes which by composition give rise to each of the classes under consideration in a single definite manner." Though Schering's proof contained not a word about groups, yet it is not essentially connected with form classes, and translated into group language. Schering's results are nothing else than those published by Kronecker.

The author next gives his reasons for excluding the consideration of the division of classes into genera, and also of Gauss's second proof of the law of reciprocity, as well as that of Legendre and the first of Kummer, from a work confined to the *elements* of the theory of numbers, while Kummer's second proof, based only on the solution of the Pellian equation and the composition of forms, closes the section. These proofs all belong to the second category, mentioned before.

An excellent table of contents facilitates reference, and the presswork done by the house of publication is so well known that nothing need be said on this head. Spaced type has been introduced occasionally to make important theorems prominent, but more might have been done to assist the reader through the eye by means of the typography.

Though dealing with the elements of the subject, the book is not well adapted to serve as a first introduction because of the unhesitating use, wherever preferable from a theoretic point of view, of more abstract methods than would really be necessary. But precisely this commends it to one with some previous acquaintance with the subject. The work is a pleasing and welcome addition to the literature of the theory of numbers, on account of its contents and their manner of presentation, but still more as the forerunner of other volumes written in the same style and with the same aims, and dealing with the less elementary portions of the subject. There exists now quite a number of text-books covering the elementary field, but a clear, well-digested presentation in coördinated and unified form, brought down to the present time, of the principal results in the field beyond, is a real want. Our author proposes a continuation of his work : that it will deserve the characterization just written, the present volume gives reason to believe. It is to be hoped that he will be able speedily to bring this difficult task to a satisfactory conclusion.

J. W. A. YOUNG.

THE UNIVERSITY OF CHICAGO, *May* 1, 1894.

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* *Gött. Abh.*, 1868.

MEMOIRS ON INFINITE SERIES.

Memoirs on Infinite Series. Published by the Tokio Mathematical and Physical Society. Tokio, Japan, 1891. 258 pp.

THIS work occupies quite a unique place among translations, and deserves a brief mention in the pages of the BULLETIN. The scope of the book is fully indicated by the table of contents, as follows:

LEJEUNE-DIRICHLET.—On the convergency of the trigonometrical series which serves to represent an arbitrary function between given limits. (Translated from French into English by R. Fujisawa.)

LEJEUNE-DIRICHLET.—On the series whose general term involves two angles and which serves to represent an arbitrary function between given limits. (Translated from French into English by R. Fujisawa.)

ABEL.—Researches on the series

$$1+\frac{m}{1}x+\frac{m(m-1)}{1\,.\,2}x^2+\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}x^3+\ldots$$

(Translated from French into English by K. Miwa.)

GAUSS.—General examination of the infinite series

$$1+\frac{\alpha\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}x^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$$

(Translated from Latin into English by D. Kikuchi.)

KUMMER.—On the hypergeometric series

$$1+\frac{\alpha\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}x^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots$$

(Translated from German into English by H. Nagaoka.)

All the above papers are from *Crelle's Journal* except the fourth, which is from Gauss's Gesammelte Werke. Of the papers themselves nothing need here be said. They are all classical works of the first importance, and are familiar to every mathematician who has a reading knowledge of French, German, and Latin; but they are now for the first time rendered accessible to English and American students who are not familiar with these languages. I have compared these translations with the original texts, and have found them literal and accurate. Our Japanese co-laborers deserve the thanks of every English-speaking mathematician for the preparation and publication of this important translation.

LEVI L. CONANT.

WORCESTER POLYTECHNIC INSTITUTE.

CORRECTION OF AN ERROR IN SALMON'S "GEOMETRY OF THREE DIMENSIONS."

THE general accuracy of Dr. Salmon's mathematical books seems to make it desirable to call attention to an error in the fourth edition of his *Geometry of Three Dimensions*. The statement "There is one pinch-point on the conic and two on the line," page 521, lines 1 and 2, should read, "There is one pinch-point on the line and two on the conic."

This interchange of "line" and "conic" may be purely a typographical error, but it has gone into Fiedler's German translation uncorrected, and is liable to cause confusion and difficulty of geometric conception.

The variety of the quartic scroll there considered (Salmon's Case IX, Cayley's Species XI, Cremona's Species IV) is the surface generated by two projectively related sheaves of planes of the second order, which are so situated as to determine a line congruence of the first order and the second class. One pair of corresponding planes of these sheaves intersect in the singular line of the congruence, so that this line is a generator and at the same time a director line of the surface. The plane of the singular conic of the congruence is excluded from the generating sheaves.

The more general surface, Salmon's Case VIII, has two pinch-points on the director conic and two on the director straight line. If the two pinch-points on the straight line coincide, while those on the conic remain distinct, either real or imaginary, we have Case IX; but if the two pinch-points on the conic coincide, the generating sheaves include the plane of the conic as a self-corresponding plane, and the surface degenerates into a ruled surface of the third order and the plane of its director conic.

These special cases were more fully discussed in my paper "On certain ruled surfaces of the fourth order," in the *American Journal of Mathematics*, volume 15.

THOMAS F. HOLGATE.

NORTHWESTERN UNIVERSITY, *April* 28, 1894.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, May 5, at half-past three o'clock, the president, Dr. McClintock, in the chair. A communication was read from the Citizens' Committee of Brooklyn inviting the members of the Society to share in the hospitalities to be extended to the American Association for the Advancement of Science on the occasion of its meeting, August 15–24, 1894. The constitution was amended in accordance with the recommendations made by the Council, the amendments to go into effect July 1, 1894. Action upon changes proposed in the by-laws was deferred until the next meeting.

THE change in the name of the Society from the NEW YORK MATHEMATICAL SOCIETY to the AMERICAN MATHEMATICAL SOCIETY will take effect July 1, 1894. A pamphlet containing the list of members, the constitution, and by-laws will be issued at about the same date. T. S. F.

MR. POMEROY LADUE, instructor in mathematics in the University of Michigan, has been appointed professor of mathematics in the University of the City of New York.

A. Z.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BECKER (H.). Die geometrische Entwickelung des Infinitesimalbegriffs im Exhaustionsbeweise bei Archimedes und ihre Bedeutung für die Differentialgeometrie und die Schule. Insterburg, 1894. 4to. 26 pp. Mk. 1.50

BOIRAC (A.). De spatio apud Leibnitium. [Thèse.] Paris, Alcan, 1894.

BURALI-FORTI (C.). Sulle classi derivate à destra e à sinistra. Torino, Clausen, 1894. (Atti della R. Accademia delle scienze di Torino, Vol. 29.) 8vo. 16 pp.

CAHEN (E.). Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues. [Thèse.] Paris, Gauthier-Villars, 1894. 4to. 97 pp.

CARLINI (L.). Saggio d'una teoria generale delle progressioni aritmetiche. Treviso, Zoppelli, 1893. 8vo. 28 pp.

CREMONA (L.). Elements of projective geometry. Translated by C. Leudesdorf. Second edition, Oxford, Clarendon Press, 1894. 8vo, 324 pp. Illustrated. 12s. 6d.

Eggenberger (J.). Beiträge zur Darstellung des Bernoulli'schen Theorems der Gammafunktion und des Laplace'schen Integrals. [Diss.] Bern, 1893. 8vo. 74 pp.

Firmicus Maternus (J.). Matheseos libri VIII, primum recensuit C. Sittel. (In 2 partibus.) Pars I: Libri I–IV. Lipsiæ, Teubner, 1894. 8vo. 15 and 246 pp. Mk. 2.40

Glaser (W.). Ueber die Ableitung der Elemente eines Kegelschnittes aus den Koefficienten der homogenen Gleichung zweiten Grades. [Progr.] Homburg, 1894. 4to. 15 pp.

Grévy (A.). Étude sur les équations fonctionnelles. [Thèse.] Paris, Gauthier-Villars, 1894. 4to. 107 pp.

Hermes (O.). Ueber Anzahl und Form von Vielflachen. Berlin, 1894. 4to. 1 plate. Mk. 1.20

Hüniger (H.). Der Philosoph Karl Christian Friedrich Krause als Mathematiker. [Progr.] Eisenberg, 1894. 4to. 32 pp. Illustrated.

Jahrbuch über die Fortschritte der Mathematik. Begründet von C. Ohrtmann, herausgegeben unter Mitwirkung von F. Müller und A. Wangerin von E. Lampe. Band XXIII: Jahrgang 1891 (in 3 Heften). Heft 1. Berlin, Reimer, 1894. 8vo. 4 pp. and pp. 1–528. Mk. 13.00

Laplace. Œuvres complètes, publiées sous les auspices de l'Académie des sciences par les secrétaires perpétuels. (En 13 vols.) Vol. X. Paris, Gauthier-Villars, 1894. 4to. 429 pp. 1 plate. Fr. 25.00

Macfarlane (A.). The principles of elliptic and hyperbolic analysis. Boston, Norwood Press [1894]. 8vo. 48 pp. $0.50

Mehler (F. G.). Ueber den Coordinatenbegriff und einige Grundlehren von den Kegelschnitten. Berlin, 1894. 8vo. 31 pp. Illustrated. Mk. 0.20

Misrachi. See **Wertheim.**

Prætorius (J.). Der Koordinatenbegriff und einige Grundlehren von den Kegelschnitten. [Progr.] Konitz, 1894. 4to. 25 pp. 2 plates.

Reynès (P.). Théorie sur la roulette. Combinaison de d'Alembert sur les chances simples; nouvelles combinaisons, dites de d'Alembert, sur les douzaines; généralisation de la théorie de d'Alembert sur une combinaison quelconque de numéros. Perpignan, Latrobe, 1894. 16mo. 20 pp. Fr. 1.50

Schlegel (V.). Die Hauptmethoden der Grassmann'schen Ausdehnungslehre in ihrer Anwendung auf die Mechanik dargestellt. (*Civilingenieur*, Vol. 40.) 4to. 17 pp.

——— Introduction aux méthodes géométriques de H. Grassmann. (*El progreso matematico*, año II, pp. 281–302.) 8vo.

Segre (C.). Introduzione alla geometria sopra un ente algebrico semplicemente infinito. (*Annali di matematica*, Serie II, Vol. 22.) Milano, Bernardoni e Rebeschini, 1894. 4to. 4 and 102 pp.

Sievert (H.). Beiträge zur Behandlung des Umkehrproblems von Thetafunktionen, deren Charakteristiken aus Dritteln ganzer Zahlen bestehen. [Progr.] Nürnberg, 1893. 8vo. 18 pp.

Strempel (F.). Ueber ein Näherungsverfahren zur Teilung von Kreisbögen. [Progr.] Rostock, 1894. 4to. 16 pp. 1 plate.

Vasìliev (A.). Bronner and Lobachèvsky. Two episodes in the life of the first professors of the University of Kazàn. [In Russian.] Kazàn, Pechenkin, 1893. 8vo. 16 pp.

——— Nikolaì Ivànovich Lobachèvsky. Address delivered at the memorial meeting of the University of Kazàn, 22 Oct. 1893. [In Russian.] Kazàn, University Press, 1894. 8vo. 40 pp.

Wertheim (G.). Die Arithmetik des Elia Misrachi. Ein Beitrag zur Geschichte der Mathematik. Frankfurt a. M., 1893. 4to. 42 pp. Mk. 1.60

II. ELEMENTARY MATHEMATICS.

Adam (V.). Taschenbuch der Logarithmen. 20ste Auflage. Wien, 1894. 12mo. 9 and 99 pp. Cloth. Mk. 1.20

Albrecht (G.). Adam Ries und die Entwicklung unserer Rechenkunst. Prag, 1894. 8vo. 18 pp. Mk. 0.40

Arzelà (C.). Complementi di algebra elementare, ad uso del secondo biennio degli istituti technici. Firenze, Le Monnier, 1894. 16mo. 125 pp. Fr. 2 50

Barlow (C. W. C.) and Bryan (G. H.). Matriculation model answers in mathematics; the examination papers from June, 1888, to January, 1894, with solutions. London, Clive, 1894. 2s.

Bernardi (G.). Soluzionario degli esercizî di trigonometria piana contenuti nel Trattato di trigonometria rettilinea e sferica di G. A. Serret (traduzione di E. Ferrucci). Firenze, Le Monnier, 1894. 16mo. 110 pp. Illustrated. Fr. 1.15

Bertrand (J.) et Garcet (H.). Traité d'algèbre. 2e partie. Nouvelle édition. Paris, Hachette, 1894. 8vo. 392 pp. Fr. 5.00

Bremiker (C.). Tafeln vierstelliger Logarithmen. 2te Auflage, besorgt von A. Kallius. Berlin, Weidmann, 1894. 8vo. 64 pp. Mk. 0.60

C. (F.). Premiers éléments d'algèbre, avec exercices et problèmes. 3e édition. Rennes, Oberthür, 1894. 12mo. 72 pp.

Chailan (E.). Résumé d'algèbre élémentaire. Tours, Mame, 1894. 18mo. 72 pp. (Alliance des maisons d'éducation chrétienne.)

Egger (J.). Uebungsbuch für den geometrischen Unterricht an Sekundarschulen und anderen Schulanstalten. (In 4 Teilen.) 2ter Teil: I. Planimetrie. 3te Auflage. Bern, Wyss, 1894. 8vo. 4 and 74 pp. Boards. Mk. 1.00

Focke (M.) und Krass (M.). Lehrbuch der Geometrie, zum Gebrauche an Gymnasien, Realgymnasien und anderen höheren Lehranstalten. 2ter Teil: Stereometrie, nebst einer Sammlung von stereometrischen Uebungslehrsätzen, Konstruktions-Aufgaben und Berechnungs-Aufgaben. 6te Auflage. Münster, Coppenrath, 1894. 8vo. 4 and 104 pp. Illustrated. Mk. 1.40

——— Lehrbuch der ebenen Trigonometrie, nebst den Grundzügen der sphärischen Trigonometrie, zum Gebrauche an Gymnasien, Realgymnasien und anderen höheren Lehranstalten. 7te Auflage. Münster, Coppenrath, 1894. 8vo. 4 and 80 pp. Illustrated. Mk. 1.20

Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 42ste Auflage. Halle, 1894. 8vo. 166 and 35 pp. Cloth. Mk. 2.50

Grubb (A. G.). Practical plane and solid geometry (elementary stages). With notes and answers for five years, 1889–1893. Subject I. London, Moffatt,, 1894. 8vo. 6d.

Hochheim (A.). Leitfaden für den Unterricht in der Arithmetik und Algebra an höheren Lehranstalten. 1tes Heft. 5te Auflage. Berlin, Mittler, 1894. 8vo. 6 and 255 pp. Mk. 3.00

Kambly's Elementar-Mathematik. (In 4 Teilen.) 2ter Teil: Planimetrie, durchgesehen und mit Uebungsaufgaben und zwei Anhängen (Trigonometrische und stereometrische Lehraufgabe der Unter-Sekunda) versehen von H. Roeder. 100ste Auflage. Breslau, Hirt, 1894. 8vo. 188 pp. Illustrated. Mk. 1.65

Loney (S. L.). Plane trigonometry. Part II: Analytical trigonometry. Cambridge University Press (New York, Macmillan), 1894. 12mo. 480 pp. Cloth. $1.00

Maupin (G.). Exercices d'algèbre à l'usage des élèves de mathématiques spéciales. Paris, Noblet, 1893. 8vo. 91 pp.

Pitz (H.). Vierstellige Logarithmentafeln. Giessen, Roth, 1894. 12mo. 16 pp. Mk. 0.40

Rankin (T. T.). Complete solutions to papers in mathematics. Second stage. Science and art examination, 1887–93. London, Moffatt, 1894. 8vo. 84 pp. 1s.

Smith (J. H.). Key to elementary algebra. New and revised edition. London, Longmans, 1894. 8vo. 396 pp. 9s.

Struve (K.). Elemente der Mathematik. Teil 1: Geometrie der Ebene. 2te Auflage. Berlin, Parey, 1894. 8vo. 8 and 59 pp. Illustrated. Mk. 0.75

Tiberi (E.). Elementi di geometria piana, ad uso degli alunni delle scuole tecniche, normali e ginnasiali. Torino, Paravia, 1894. 16mo. 7 and 157 pp. Fr. 2.50

Walsh (J. H.). Mathematics for common schools. 3 parts. Boston, Heath, 1894. 8vo. $1.15

III. APPLIED MATHEMATICS.

Autenrieth (E.). Die statische Berechnung der Kuppelgewölbe. Berlin, Springer, 1894. 8vo. 4 and 75 pp. 15 plates. Mk. 4.00

Bagard (H.). Sur les forces électromotrices thermo-électriques et le transport électrique de la chaleur dans les électrolytes. [Thèse.] Paris, Gauthier-Villars, 1894. 4to. 59 pp. Illustrated.

Baule (A.). Détermination de la vitesse d'un navire par les ondes qu'il soulève; effets produits par celles-ci sur les lochs mécaniques. Paris, 1894. 8vo. Fr. 2.50

Bedell (F.) and **Crehore** (A. C.) Alternating currents: an analytical and graphical treatment for students and engineers. 2d edition. New York, Johnston, 1894. 8vo. 324 pp. $2.50

Bouasse (H.). Réflexion et réfraction dans les milieux isotropes, transparents et absorbants. Paris, 1892. 8vo. 158 pp. Illustrated. Fr. 7.50

Du Bois (H.). Magnetische Kreise, deren Natur und Anwendung. Berlin, Springer, 1894. 8vo. 14 and 382 pp. Illustrated. Cloth. Mk. 10.00

Dühring (E.). Critical history of the general principles of mechanics. Russian translation by N. Marakùiev. Moscow, 1893. 8vo. 551 pp. Mk. 12.00

Emtage (W. T. A.). An introduction to the mathematical theory of electricity and magnetism. 2d edition, revised. Oxford, Clarendon Press, 1894. 8vo. 264 pp. 7s. 6d.

Engelhardt (M.). Copernicus, his life and works. (In Russian.) St. Petersburg, 1891. 8vo. 69 pp. Mk. 1.50

Fortschritte der Physik im Jahre 1888. Dargestellt von der Physikalischen Gesellschaft zu Berlin. Jahrgang 44 (in 3 Abteilungen). Abt. I: Physik der Materie, dargestellt von R. Börnstein. Braunschweig, Vieweg, 1894. 8vo. 62 and 479 pp. Mk. 20.00

Germán y Moreno (J.). Nociones elementales de mecánica racional y aplicada. Cuaderno I: Estática, analítica y gráfica, y resistencias pasivas. Madrid, 1894. 4to. 156 pp. Illustrated. Mk. 3.50

Gonier (A.). Traité raisonné de perspective linéaire, à l'usage des instituteurs. Dijon, Union typographique, 1893. 8vo. 8 and 135 pp.

Graf (E.). Die Theorie der Akustik im griechischen Altertum. Leipzig, 1894. 4to. 16 pp. Mk. 1.20

Gran (A.). Tabeller til brug ved bestemmelse af maanens deklination og höide. Christiania, 1894. 8vo. 8 pp. Mk. 0.80

Guilhon (E.). Théories météorologiques et prévision du temps. Paris, 1894. 8vo. 96 pp. Illustrated. Fr. 4.50

Gyldén (H.). Traité analytique des orbites absolues des huit planètes principales. Vol. 1: Théorie générale des orbites absolues. Stockholm (Paris, Hermann), 1894. 4to. 8 and 577 pp. Fr. 35.00

Heath (R. S.) Lehrbuch der geometrischen Optik. Deutsche Ausgabe von R. Kanthack. Berlin, Springer, 1894. 8vo. 13 and 386 pp. Illustrated. Mk 10.00

Klumpke (D.). Contribution à l'étude des anneaux de Saturne. Paris, Gauthier-Villars, 1894. 4to. 70 pp. Illustrated.

Martin (T. C.). The inventions, researches, and writings of Nikola Tesla: with special reference to his work in polyphase currents and high-potential lighting. New York, *The Electrical Engineer*, 1894. 8vo. 496 pp. With portrait and illustrations. $5.00

Messerschmitt (J. B.). Lotabweichungen in der Westschweiz. (Das Schweizerische Dreiecksnetz der internationalen Erdmessung, Band 6.) Zürich, Fäsi und Beer. 4to. 4 and 200 pp. 1 plate. Mk. 10.00

Miething (E.). L. Euler's Lehre vom Aether. Berlin, 1894. 4to. Mk. 1.00

Müller-Pouillet. Lehrbuch der Physik und Meteorologie. 9te, umgearbeitete und vermehrte Auflage von L. Pfaundler, unter Mitwirkung von O. Lummer. (In 3 Bänden.) Band II: Optik und Wärmelehre. Abteilung 1, Lieferung 1. Braunschweig, Vieweg, 1894. 8vo. Mk. 4.00

Planck (M.). Heinrich Rudolf Hertz. Rede zu seinem Gedächtnis in der Sitzung der Physikalischen Gesellschaft zu Berlin am 16. Februar 1894. Leipzig, Barth, 1894. 8vo. 23 pp. Mk. 0.60

Rankin (T. T.). Applied mathematics, advanced stage. With answers to the calculations. London, Simkin, 1894. 12mo. 30 pp. 4d.

Sparre (M. de). Sur le mouvement des projectiles oblongs autour de leur centre de gravité et sur les conditions de stabilité de ces projectiles. Nancy, 1894. 8vo. 58 pp. Fr. 2.50

Thwing (C. B.). Bestimmung der Dielectricitätsconstante mittels der Resonnanz. [Diss.] Bonn, 1894. 8vo. 21 pp. 2 plates.

Tischner (A.). Le système solaire se mouvant. Leipzig, Fock, 1894. 8vo. 19 pp. Mk. 0.80

Uckermann (K.). Ueber das Princip der kleinsten Wirkung. [Diss.] Marburg, 1893. 8vo. 41 pp.

Van der Waals (J. D.). La continuité des états gazeux et liquides. Traduit de l'allemand et annoté par Dommer et Pomey, avec une préface de Sarrau, et suivi de notes sur les travaux de Ph. Guye et Mathias. Paris, Carré, 1894. 8vo. 16 and 280 pp.

Velde (W.). Die magnetischen Kraftlinien im physikalischen Unterricht. Berlin, 1894. 4to. 2 plates. Mk. 1.20

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LOBACHÈVSKY AS ALGEBRAIST AND ANALYST.

BY PROF. A. VASILIEV.

THE mathematical genius of Lobachèvsky manifested itself not in geometry alone. His early study of Gauss's *Disquisitiones arithmeticæ*, under the direction of Professor Bartels, led him in 1813 to write a memoir (which has never been published and seems to be lost) on the resolution of the binomial equation $x^n - 1 = 0$ in the case $n = 4p + 1$. At a later period, in 1834, he returned to these studies and carefully examined the case $n = 8p + 1$ (see his paper "Reduction of the degree of the binomial equation when the exponent, diminished by 1, is divisible by 8," written in Russian and published in the first volume (1834) of the Scientific Memoirs of the University of Kazàn).

Next to geometry, Lobachèvsky's favorite subject was the systematic exposition of algebra. He considered algebra as intended to lay the rigorous foundations for mathematical science, and this idea he carried out in a work published in 1833 under the title "Algebra, or calculus of finite quantities" (Russian). This extensive work is remarkable alike for the rigor of its definitions and proofs, and for the width of its scope. Thus we find here treated not only the solution of numerical equations and Gauss's theory of the resolution of the equation $x^n - 1 = 0$, but also the calculus of finite increments. Through his geometrical researches Lobachèvsky had been led to the necessity of defining the trigonometric functions independently of all geometric considerations; this idea he introduced into his algebra, in which the trigonometric functions cos z and sin z are defined by means of the series $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$ and $\frac{z}{1} - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$, and all their properties are derived from this definition. I believe that Lobachèvsky was the first to expound systematically the theory of trigonometric functions on this basis.*

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* Professor A. Macfarlane in his interesting paper "On the definitions of the trigonometric functions" mentions De Morgan as expressing the same idea in his "Trigonometry and Double Algebra." Professor Mansion attributes the idea to Mr. Seidel (*Crelle's Journal für Math.*, vol. 73, 1871).

[This must be a misunderstanding, as Seidel's paper deals with products and not with series. Cauchy, in his "Analyse algébrique," 1821, p. 309, distinctly defines sin z and cos z by means of the infinite series. —A. Z.]

This point of view naturally led Lobachèvsky to the question of the convergence of infinite series; he invented a special method for testing the convergence of a series and explained it in several memoirs:

(1) "On the convergence of trigonometric series" (Russian), in the *Scientific Memoirs of the University of Kazàn*, vol. I, 1834;

(2) "A method for ascertaining the convergence of infinite series and for approximating the values of functions of very large numbers," ib., vol. II, 1835;

(3) "Ueber die Convergenz der unendlichen Reihen," Kasan, 1841, 4to.

Lobachèvsky's method is based on the following considerations. Let there be given an infinite series

$$\sum_{i=1}^{i=\infty} f(i) = f(1) + f(2) + \ldots + f(i) + \ldots; \qquad (1)$$

every term of this series can evidently be expanded as a fraction in the form

$$\frac{\lambda_1}{2} + \frac{\lambda_2}{2^2} + \frac{\lambda_3}{2^3} + \ldots,$$

where each λ is equal to 1 or to 0.* Suppose now we have found a number μ such that

$$f(\mu) \geqq 2^{-\lambda} \text{ and } f(\mu + 1) < 2^{-\lambda}; \qquad (2)$$

then not more than μ terms can have the fraction $1/2^\lambda$ in their expansion, and in the sum of the series this fraction $1/2^\lambda$ cannot have a coefficient exceeding μ.

It follows that the sum of the series cannot exceed $\sum_{\lambda=0}^{\lambda=\infty} \mu\, 2^{-\lambda}$. Thus an upper limit is found for the sum of the series, and this may lead to the determination of its convergence. The difficulty lies in determining μ so as to satisfy the inequalities (2).

It is worthy of notice that Lobachèvsky applied this method, based on the determination of an upper limit of the sum, even to the simple exponential series $\Sigma \frac{x^i}{i!} = f(x)$, for the purpose of proving the property

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* This is merely an adaptation of the idea of decimal fractions to the binary system of numeration.

$$f(x) \,.\, f(y) = f(x + y).$$

He seems to have been guided here by the very considerations which, later on, led mathematicians to the distinction between uniform and non-uniform ("gleichmässige" and "ungleichmässige") convergence.

There can certainly be no doubt but that the keen genius of Lobachèvsky perceived the insufficiency of the assumption that every continuous function is necessarily capable of differentiation. It is well known that this assumption was made by all mathematicians until Weierstrass showed it to be untrue, by the example of the function

$$\sum_{n=0}^{n=\infty} b^n \cos \pi(a^n x)$$

(where a is an odd integer, b a positive proper fraction, and $ab > 1 + \frac{3}{2}\pi$), which though continuous is not "*differentiirbar.*" *

Lobachèvsky developed his ideas on this subject with particular clearness in his Russian memoir of 1835 ("A method for ascertaining, etc."), in which he says: "*The function is gradual* (postepènna) *if its increment diminishes to zero along with the increment of the variable x. The function is continuous* (neprerўvna) *if the ratio of these increments, as the latter diminish, passes into a new function, which is the differential coefficient.* Integrals must always be divided into intervals in such a manner that the elements remain both gradual and continuous."

These words clearly show that Lobachèvsky was far in advance of his contemporaries in this question concerning the principles of the infinitesimal calculus, just as he was in advance of them on the foundations of geometry.

He had expressed these views more elaborately in his memoir of 1834 ("On the convergence of trigonometric series"). Here he gives his definition of the differential coefficient in the following terms: "Let $F(x)$ denote a function which, varying with x, increases from a certain value of x to $x = a$. We divide $a - x$ into i equal parts and put $(a - x)/i = h$. Let the quantities

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* This example was first published by P. du Bois-Reymond in his "Versuch einer Classification der willkürlichen Functionen" (*Crelle's Journal für Mathematik*, vol. 79).

$$F(x), \quad F(x+h), \quad F(x+2h), \quad \ldots \quad F(a)$$

be known for every small value of h, which decreases indefinitely as i increases. The ratio $\dfrac{F(x+h) - F(x)}{h}$ will vary with h. For $i' > i$, let $(a-x)/i'$ be equal to h'. If now the difference

$$\frac{F(x+h) - F(x)}{h} - \frac{F(x+h') - F(x)}{h'} = \epsilon,$$

or, which amounts to the same,

$$\frac{h'F(x+h) - hF(x+h') + (h - h')F(x)}{hh'} = \epsilon,$$

decreases simultaneously with h for every value of x and can be made as small as we please, the function $F(x)$ receives the name of a *continuous* function. The ratio $\dfrac{F(x+h) - F(x)}{h}$ has in this case a limit which is obtained by letting h decrease; and this limit is $\dfrac{dF(x)}{dx}$."

These are Lobachèvsky's ideas concerning the foundations of the differential calculus. As regards the integral calculus, it is well known that Lobachèvsky repeatedly insisted on the great value of his imaginary geometry for the evaluation of integrals. (See *N. I. Lobatcheffsky, Collection complète des œuvres géométriques*, vol. II, p. 613, 655 sq.) In 1836 he published a paper (in Russian) in the *Memoirs of the University of Kazàn* entitled "Application of the imaginary geometry to some integrals" (see ib., vol. I, pp. 121–218). This extensive memoir contains many results of great value in the integral calculus. Lobachèvsky insists in particular on the importance of the integral $L(x) = -\int_0^x dx \log \cos x$, to which a large number of very complicated integrals are reducible. Lobachèvsky's work in the integral calculus is still awaiting a full and careful appreciation and may perhaps be made the subject of another paper.

In conclusion I may perhaps be allowed to express the hope that the appreciation of the work of our great Russian mathematician, manifested by the mathematicians assembled at the

Chicago Congress through their generous contribution to the Lobachèvsky memorial fund, contains the promise of a still closer union between the mathematicians of America and Russia, and proves the solidarity of scientific interests among all nationalities.

KAZÀN, *March* 7, 1894.

MACFARLANE'S ALGEBRA OF PHYSICS.

Principles of the Algebra of Physics. By A. MACFARLANE, Professor of Physics in the University of Texas, Austin, Texas. *Proceedings of the American Association for the Advancement of Science*, vol. 40, 1891, 53 pp.

On the Imaginary of Algebra. By A. MACFARLANE. *Ibid.*, vol. 41, 1892, 23 pp.

THE purpose of the first of the articles which are to form the subject of this review may most properly be stated in the author's own words: "The guiding idea in this paper is generalization. What is sought for is an algebra which will apply directly to physical quantities, will include and unify the several branches of analysis, and when specialized will become ordinary algebra."

A student who sets out to use Grassmann's algebra in geometrical work finds that it applies beautifully to projective problems in curves and surfaces of no higher order than the second, but beyond them he is confronted and stopped by difficulties which can be overcome only by the study of the ordinary theory of algebraic forms. In the same way quaternions work out many metrical properties of curves and surfaces with facility and grace, but I think every student who has tried to go far with them finds that he is at last brought back to the study of the equations and functions of ordinary analysis. There seems to be no way around the difficulties of the theories of forms and functions, and even when results have been attained by methods which appear to avoid them the mind is seldom convinced of their validity. As we shall see, Professor Macfarlane derives the formulas of trigonometry with great facility, but it seems almost certain that no analyst would dare to use them if they had no other foundation.

Passing by considerations of this kind which seem to make it doubtful whether or not any system of analysis other than the ordinary one can do much to advance mathematical science, we come to the author's first objection to quaternions as Hamilton left them. Hamilton made the square of a vector negative; in fact he wrote

$$ij = k$$

and

$$i.ij = ik = -j;$$

so that he had to take $i^2 = -1$ if he made

$$i(ij) = (ii)j = i^2j;$$

that is, if he preserved the association law in multiplication. The result of this is that the scalar parts of Hamilton's products are taken negatively. Professor Macfarlane prefers to make them positive even at the expense of the associative law. He finds

(1) That problems in vector analysis can be worked out without the minus;

(2) That the expressions so obtained are more consistent with those of algebra;

(3) That the square of a directed quantity in algebra is positive;

(4) That the quaternion rule throws the system out of harmony with determinants;

(5) That it makes the operator ∇^2 negative.

As to the third of these points the rule stated is certainly very limited; while $(-3)^2 = 9$, it is also true that $(3\sqrt{-1})^2 = -9$, and $3\sqrt{-1}$ is just as much a directed quantity as -3; but, granting them all, one is hardly convinced that Hamilton did not choose wisely in preferring the conservation of the associative law. Professor Macfarlane's own ingenious equations show how inconvenient analysis becomes without it. To be sure, each equation means more, but the question is whether it is better to manipulate two or three facile equations or a single "anfractuous" one.

Proceeding with his argument for making the square of a vector positive, he finds what amounts to a contradiction in Hamilton's system when vectors are taken to represent physical entities. The units i, j, k admit of a double interpretation: (1) each causes rotation through the angle $\frac{1}{2}\pi$ in the plane to which it is at right angles; (2) they are simply directed lines. In the first case we may find the meaning of ij by adding the quadrants which represent their successive effects, and it comes out easily that ij must equal k, while ii represents the rotation π, or simply -1. Now if i is a rotator and j a line, we find as before that $ij = k$, where k is a line; while

ii, where the first i is a rotator and the second a line, ought to leave the line i unchanged. As a matter of fact in quaternions $ii = -1$ always. There is certainly what seems an "insuperable objection," but from Hamilton's point of view it is no objection at all. Hamilton lays down a consistent body of symbols and combinatory laws, and then makes such physical applications as he can. The fact that the symbols fail of certain representative powers is no objection to them as a logical system. The truth is that a vector symbol will not bear interpretation as a rotator except when it is multiplied into a vector perpendicular to itself, and then it will bear that interpretation. This is a limitation, but it is not an objection. The laws of multiplication are first laid down and then we ask for their physical meaning; the other course might be taken and Professor Macfarlane actually takes it, but it is hard to see why one of them is more right or wrong than the other. All through his discussion of Hamilton's rules the author seems to identify the rotator i (or $i^{\frac{\pi}{2}}$) with the quadrant corresponding; this introduces difficulties which are perhaps unnecessary. Hamilton looked upon the arc rather as the logarithm of the rotator, and these arcs have in fact a remarkable resemblance to ordinary logarithms. Their addition is not commutative, but the addition of the logarithms of rotators should not be.

Leaving this interesting subject, Professor Macfarlane asks next, "Are the principles of the method of quaternions consistent with the theory of dimensions . . . ?" If i and j denote lengths, then in the equation

$$ij = k$$

k ought to denote an area. This is undoubtedly true, and in any equation where i and j were interpreted as lines k would have to be interpreted as an area; and to this interpretation the rules of quaternions offer no hindrance. Directed areas are a part of the system. The difficulty about dimensions is purely psychological and comes from confusing in thought the symbol with what it may, but need not invariably, stand for. The symbols i, j, k denote imaginary numbers; for these numbers there is an arithmetic, and one of the rules of this arithmetic is that when taken abstractly ij shall give k, where k is the same kind of a thing as i and j; just as 3 times 2 gives 6. The dimension theory is not in question. If the equations are to represent truths of geometry they must be homogeneous, but the homogeneity takes care of itself generally. It is perfectly useless to try to take actual lines and manipulate them mathematically; between the symbol and the reality there is an impassable gulf. The equation cited by Professor Macfarlane from the "Directional Calculus"

$$p_2 = p_1 + \epsilon,$$

where p_2 and p_1 denote points and ϵ a line, is necessarily a pseud-equation as he implies, and results apparently obtained from it can only be delusive unless the so-called points are really lines; but quaternions present no such anomalies. It seems to be unnecessary to try to make i, j, k mean merely lines or merely directions, or to lay down such a law as that "in such an expression as xi it is more philosophical and correct to consider x as embodying the unit, while i denotes simply the axis." It is surely not a defect of Hamilton's system that x may or may not be taken as a pure tensor according to circumstances, and that the number i may be of one or two dimensions in the unit of length like any other number, or that it need not contain the unit of length at all but may denote a certain rotation or a certain axis of rotation.

The essential difference between Professor Macfarlane's system of algebra and that of Hamilton is that he makes $i^2 = j^2 = k^2 = +1$ instead of -1. This destroys the associative character of the multiplication, but it allows an extension to manifoldnesses which quaternions cannot reach. With this rule the product of the imaginary numbers $A = ai + bj + ck$ and $B = li + mj + nk$ easily takes the form

$$AB = \cos AB + \text{Sin } AB,$$

where the meaning of the symbols cos and Sin is pretty obvious. $\cos AB = - \text{S}.AB$ in the language of quaternions and $\text{Sin } AB = \text{V}.AB$. The analytical meanings of cosine and sine are so well established that one would be tempted to wish the author had not made this use of them, especially as his reasons for discarding Hamilton's expressive symbols would apply quite as well to his own.

The formulas for the products of three and four of these numbers grow complex and would seem difficult to employ. Since $ABCD$ may mean various things according to the mode of associating the letters, it is indispensable to use the greatest care at every step. The author selects from the five possible products the two $((AB)C)D$ and $(AB)(CD)$ as the most important. The form $(AB)(CD)$ differs from the product of two quaternions on account of the squares of the units being positive.

A quaternion is represented by the symbol $a\alpha^A$, where a is the tensor, α the axis, and A the angle. The ordinary representation is $a\alpha^{\frac{2A}{\pi}}$, so that here α^A is the same as the familiar

$\alpha^{\frac{2A}{\pi}}$, and it comes out that α^{π} is the old α^2 or -1. Of course Professor Macfarlane's α^2 denotes a rotation through the angle 2 in circular measure, and there is no difficulty in distinguishing between this symbol and the α whose square is $+1$ in his system. Remembering that α^A turns a line lying in a plane at right angles to α through the angle $\frac{\pi}{2}$ it is evident that

$$a\alpha^A = a\,(\cos A + \sin A \,.\, \alpha^{\frac{\pi}{2}}),$$

as the author asserts. This is Hamilton's equation :

$$a\alpha^{\frac{2A}{\pi}} = a\,(\cos A + \sin A \,.\, \alpha),$$

the difference being merely one of notation. This new $\alpha^{\frac{\pi}{2}}$ exactly places the quaternion unit α in a multiplication-table. We may call the part in the parenthesis a versor without prejudice to either system, and by forming the product of two versors and taking its scalar part we get the fundamental formula

$$\cos C = \cos A \cos B - \sin A \sin B \cos C.$$

It is well known that the algebra of quaternions all having the same plane is not different from that of ordinary complex quantities, except in form. Thus we may write with Professor Macfarlane

$$\alpha^A = \cos A + \alpha^{\frac{\pi}{2}} \sin A,$$

or, as usual,

$$e^{iA} = \cos A + i \sin A,$$

and proceed to develop a plane trigonometry. The analyst will be likely to prefer the latter because he can reach it step by step from the multiplication-table he learned at school with no breach in the continuity of his logic, and with no hypothesis as to the geometrical meaning of i. Moreover the second equation is true in the same way that $3 + 2 = 5$ is true. It does indeed define the function e^{iA}, but perhaps in a very different sense from that in which the first equation may be said to define α^A. Changing A to $A + B$ and factoring the right member, we easily conclude with our author that

$$\alpha^{A+B} = \alpha^A \,.\, \alpha^B,$$

with commutative factors, and De Moivre's theorem is a near consequence. We shall have $(\alpha^A)^n = \alpha^{nA}$, but it is by no means true that $(\alpha^A\beta^B)^n = \alpha^{nA}.\beta^{nB}$. The result is much more complicated, for $(\alpha^A\beta^B)^n$ is a complex number of the form $(l + m\gamma^{\frac{\pi}{2}})^n$, where only γ is an imaginary, and the binomial theorem must be used.

Remembering that $\alpha^{\frac{\pi}{2}}$ has all the analytical properties of $\sqrt{-1}$, at least when not combined with other imaginaries, Professor Macfarlane easily shows that $\alpha^A = e^{A\alpha^{\frac{\pi}{2}}}$. Of course he implicitly defines the symbol $e^{A\alpha^{\frac{\pi}{2}}}$ by the series

$$1 + A\alpha^{\frac{\pi}{2}} + \ldots$$

He then concludes without difficulty that the logarithm of the quaternion $r\alpha^A$ is $\log r + \alpha^{\frac{\pi}{2}}A$; an expression which he shows how to generalize.

The products of vectors not being associative, the problem of finding their derivatives with respect to a scalar variable presents some interesting matter; and formulas for $\frac{d}{dt}(A^n)$, where A is a vector, are derived simply and elegantly. Of course the differentiation of a quaternion presents no new principles, and the same may perhaps be said of the author's treatment of matrices; but at any rate his remarks upon the latter subject are very interesting reading. After all, however, there is a certain flavor of genius in what Hamilton and Tait have to say about this particular kind of linear substitutions which nothing more recent seems to replace exactly. They certainly have the advantage in simplicity of language and method, and they are much more easily understood.

Professor Macfarlane dwells at some length on the properties of the operator $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, which is formally a vector and shows easily that, if u is a scalar quantity,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2},$$

whereas in quaternions ∇^2 is negative.

When C is a vector ∇C will contain a scalar and vector part and he writes the solenoidal condition

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$$

in the concise form cos $\nabla C=0$; while the condition for no molecular rotation is Sin $\nabla C=0$. Tait writes these conditions:

$$\mathrm{S}.\,\nabla C=0; \quad \mathrm{V}.\,\nabla C=0.$$

Our author prefers to reserve the symbol ∇^2 for Laplace's operator and remarks that, for a vector,

$$\nabla(\nabla C) \neq \nabla^2 C.$$

Quite commonly $\nabla(\nabla o)$ would be written $\nabla^2 o$, and his remark that "$\nabla(\nabla C)$ is not equal to $\nabla^2 C$" can only mean that ∇^2 on a vector gives a result very different in form from ∇^2 on a scalar function. The first paper closes with some interesting examples of the use of ∇ and remarks on the addition of scalar and vector quantities having fixed positions in space.

In his second article Professor Macfarlane proposes "to review the different explanations [of the $\sqrt{-1}$] . . . . which have been contributed, with the hope of finding a theory which will tend to unify them." He points out that the investigation of the subject was started by the controversy about the logarithms of negative numbers and quotes from D'Alembert the following astonishing fallacy: "$e^{\frac{1}{2}}=+\sqrt{e}$ or $-\sqrt{e}$; but the logarithm of e is $\frac{1}{2}$; therefore the logarithm of $-\sqrt{e}$ as well as $+\sqrt{e}$ is $\frac{1}{2}$." Homer nods sometimes, but it is hard to imagine D'Alembert believing that by taking merely arithmetical roots of e he could produce a negative number. It recalls some of Euler's queer fancies about infinite series. After recalling various theories of $\sqrt{-1}$ our author pronounces its true explanation to be that "of a geometric ratio or quaternion" with "at least one other geometric meaning."

He treats next spherical trigonometry by quaternions. Recalling his equations

$$\alpha^A=\cos A+\alpha^{\frac{\pi}{2}}\sin A$$

and

$$\alpha^A=e^{A\alpha^{\frac{\pi}{2}}},$$

he easily concludes again that $A\alpha^{\frac{\pi}{2}}$ is the logarithm of α^A. The formal similarity of the powers of α to those of e^i being complete it can be seen at once how expressions of the most general kind can be obtained for the logarithms of negative numbers. By multiplying together α^A and β^B and taking the real and imaginary parts the fundamental equations of spherical trigonometry are reached almost instantaneously as in the former paper.

A section on circular spirals closes with the differentiation of α^A and other quaternion expressions, and with some remarks on hyperbolic trigonometry and hyperbolic spirals the article terminates. The hyperbolic trigonometry is founded on the equation

$$h\alpha^A = \cosh A + \alpha^{\frac{\pi}{2}} \sinh A,$$

where the symbols on the right have their usual meaning, and A is the area of a hyperbolic sector.

C. H. CHAPMAN.

UNIVERSITY OF OREGON, *May* 14, 1894.

NOTE ON THE SUBSTITUTION GROUPS OF EIGHT AND NINE LETTERS.

BY G. A. MILLER, PH.D.

IN calculating the possible groups of a given degree it is very helpful to have an accurate list of the groups of the lower degrees. An error in the lower groups is apt to give rise to numerous errors in the higher groups. On this account I have calculated all the possible groups through degree 9 and compared my results with the published lists. No complete lists of the groups beyond degree 9 have yet been published.

In the April number of this journal I noted several errors and one omission in the lists of the groups of eight letters. The following forms a supplement to this note.

There is a primitive group of degree 8 and order 1344, which is not given in the lists referred to in my note in the April number of this journal. The existence of a transitive group of this order and degree can be proved as follows:

$$A(abcdefgh)_8$$

consists of these substitutions:

$$\begin{array}{ll}
1 & ab\,.\,cd\,.\,ef\,.\,gh \\
 & ac\,.\,bd\,.\,eg\,.\,fh \\
 & ad\,.\,bc\,.\,eh\,.\,fg \\
 & ae\,.\,bf\,.\,cg\,.\,dh \\
 & af\,.\,be\,.\,ch\,.\,dg \\
 & ag\,.\,bh\,.\,ce\,.\,df \\
 & ah\,.\,bg\,.\,cf\,.\,de
\end{array}$$

This group evidently contains as factors of its substitutions all the cycles whose degree and order are two in the symmetric group of eight letters. We can prove that there are thirty such groups in this symmetric group. For we may suppose the first column of cycles to be fixed for all these groups; the rest of the first substitution may then be formed in

$$\frac{\frac{6\,.\,5}{2}\times\frac{4\,.\,3}{2}\times\frac{2\,.\,1}{2}}{6}=15$$

ways. Since the first two substitutions of this group are contained in the group obtained by a (1, 1) correspondence between two four-groups (Vierergruppe), the second substitution can be completed in two ways after the first has been chosen. These two substitutions with the given column of cycles determine the group. There are therefore thirty such groups in the symmetric group, and each is transformed into itself by $9!/30 = 1344$ substitutions, which constitute the transitive group in question.

Having proved the existence of the group it is necessary to show that there is only one such group and to find its generating substitutions. The group must have the group of order 168 and degree 7 as a subgroup. It therefore has 112 groups of the form

$$(abe\,.\,cdf)\ \text{cyc}$$

and $\frac{1344}{112} = 12$ positive substitutions which transform this last group into itself. Since six such substitutions, viz.,

$$\begin{array}{lll}
1 & ab\,.\,cd & abe\,.\,cdf \\
 & ae\,.\,cf & aeb\,.\,cfd \\
 & be\,.\,df &
\end{array}$$

are in the given group of order 168, the other six must be of degree 8 and must transform these six into themselves. The only substitutions which satisfy the requirements are

$$\begin{array}{ll} ac\,.\,bd\,.\,ef\,.\,gh & adecbf\,.\,gh \\ ac\,.\,bf\,.\,de\,.\,gh & afbced\,.\,gh \\ ad\,.\,bc\,.\,ef\,.\,gh & \\ af\,.\,bd\,.\,ce\,.\,gh & \end{array}$$

These must therefore be in the group. By multiplying $ac\,.\,bd\,.\,ef\,.\,gh$ into $abcefdg$ we have

$$aedcbgh.$$

Since the powers of this substitution cannot be in the given 168 there are at least 168 × 7 common substitutions in all groups of order 1344 and degree 8. Hence there can be only one such group.

From what has been said it follows directly that the group is generated by

$$(+\,abcdef\,)_{24},\quad abcefdg,\quad aedcbgh.$$

A more convenient notation is

$$+\,[\{(ab)(cd)(eh)(fg)\}\ \text{pos}\ (ABCD)\ \text{all}](abcefdg)\ \text{cyc}.$$

After having worked out this group I found it in Kirkman s list of transitive groups through degree 10, *Proceedings of the literary and philosophical society of Manchester*, vol. 3, and in Jordan's classification of primitive groups, *Comptes Rendus*, vol. 73. As very little is given in regard to the group in either of these places I have not hesitated to publish this proof of its existence which at the same time establishes the important property that it is the largest group of degree 8 which contains $A(abcdefgh)_8$ as a self-conjugate subgroup and exhibits it in the notation employed by Professors Cayley and Cole in their late lists. It needs scarcely be added that this group is threefold transitive.

The only complete list of the groups of degree 9 is that given by Professor Cole in the *Quarterly Journal of Mathematics*, vol. 26. In this list I found only two omissions, viz., the two intransitive groups of order 6, $\{(abcdef)\ \text{cyc}\ (ghi)\ \text{cyc}\}$ tris and $(ab\,.\,cd\,.\,ef)(ghi)$ cyc. The substitutions are

$$\begin{array}{llll} 1 & ad\,.\,eb\,.\,cf & abcdef\,.\,ghi & ace\,.\,bdf\,.\,gih \\ & & afedcb\,.\,gih & aec\,.\,bfd\,.\,ghi \end{array}$$

and

$$\begin{array}{llll} 1 & ab\,.\,cd\,.\,ef & ghi & ab\,.\,cd\,.\,ef\,.\,ghi \\ & & gih & ab\,.\,cd\,.\,ef\,.\,gih \end{array}$$

Jordan gives an enumeration of the primitive groups through degree 17 in *Comptes Rendus*, vol. 75, in which the number of primitive groups of degree 9 (excluding the groups that contain the alternating group) is given as eight, while Professor Cole's list contains nine such groups. The group omitted is that of order 1512, as may be learned from Jordan's article on the classification of primitive groups in volume 73 of the same journal.

By these additions the number of known groups of degree 8 becomes 200 instead of 199 as stated in my former note, and the number of groups of degree 9 becomes 258.

UNIVERSITY OF MICHIGAN, *May*, 1894.

FOURIER'S SERIES AND HARMONIC FUNCTIONS.

An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with applications to problems in Mathematical Physics. By WILLIAM ELWOOD BYERLY, Professor of Mathematics in Harvard University. Boston: Ginn & Co., 1893. 8vo, xii and 288 pp.

THIS book has recently been made the subject of a rather singular review in a leading New York paper, in which a number of curious statements are made. The reviewer begins with the statement that, "notwithstanding its name, so redolent of Helicon, there is mighty little poetry in spherical harmonics." He then, apparently overlooking the greater part of the contents of the book, and even of the title, goes on to give a rather restricted description of the use of spherical harmonics, ending up with the statement that the subject "is of great utility, and, like other utility-mathematics, is tedious, difficult, disagreeable, and unbeautiful." This is rather discouraging to one intending to read Professor Byerly's book, and, at the risk of being thought rash, I shall venture to disagree somewhat from the learned reviewer. No doubt the interest and beauty of a mathematical subject is largely a matter of personal taste, and one may profess a dislike for any subject involving the necessity of developments in infinite series, as he may to the employment of irrationals. But in regard to the subject of partial differential equations, to which this subject properly belongs, the opinions of many would be different from that above cited. The present writer has always considered this subject, upon which nearly all problems in mathematical physics depend, one of the most interesting in the range of analysis. He must even admit that, in case of fire in his library, the first books he would seize to rescue would be Kirchhoff's *Mechanik* and Riemann-Hattendorff's *Partielle Differentialgleichungen.*

The partial differential equations of mathematical physics are nearly all linear, with constant coefficients. By all odds the most important ones are, as a matter of fact, particular forms of the equation

$$a\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} + c\left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right\} = 0.$$

This includes the equations for the propagation of waves in elastic solids and fluids, and of electromagnetic disturbances, the equation for the conduction of heat and electricity, for diffusion of liquids and gases, for the vibrations of strings, membranes, and air in musical instruments, for the propagation of telegraphic and telephonic signals, and many others. The mathematical problem is in all cases of the same nature. The arbitrary functions that arise in the integration of a partial differential equation are to be determined by imposing, for a certain value of t, or at a certain surface, or both, given values upon the dependent variable u, and, if possible, upon one of its derivatives. Cauchy gave a general method of effecting the determination, depending upon a particular method used by Fourier, but it is not always convenient. The first half of Professor Byerly's book deals with Fourier's method of solution.

One of the earliest, and the most famous solution of the class here referred to, was that given by Laplace of what has since received the name of Dirichlet's Problem. If in the above equation a and b are zero, we are to determine u as a uniform continuous function within or without a certain surface by giving its values at every point of the surface. The method of Laplace, applicable when the surface is a sphere, and generalized by Lamé, was to transform the differential equation into one involving three space parameters as independent variables, of such a nature that for the surface in question one of the parameters is constant, and u is given as a function of the other two. u is then developed in a convergent series of functions of the three parameters, of such a nature that each term satisfies the differential equation, and is besides a product of three functions, each involving only one of the three variable parameters. The partial differential equation then resolves itself into three ordinary differential equations, so that the problem is much simplified. The general question, then, involves particular solutions of certain ordinary differential equations, and the development of arbitrarily given functions of one or two variables in series involving them.

In his introductory chapter Professor Byerly introduces the various equations to be treated, together with their transformations in terms of the various variables required, showing how the particular solutions, such as sines and cosines, Legendre's and Bessel's functions, arise, and illustrating the use made of them. The remainder of the first half of the book is devoted to problems solved by Fourier, where the functions used in the developments are sines and cosines of multiples of a single parameter. An elementary treatment of Fourier's series is given, which is clear enough to the student, but suffers under the disadvantage of not being rigorous. Chapter III is devoted to the subject of Fourier's series, although the frank statement is made that the subject is altogether too large to be completely handled in an elementary treatise. This seems to the writer a great pity if it is true. Professor Byerly's book is the result of a course of lectures delivered at Harvard, and the writer has scanned the Harvard catalogue in vain for an indication of a course in which the gap here left would be filled up. Dirichlet's and Riemann's work would of course be referred to, but treatments of Fourier's series in English generally leave much, or rather everything, to be desired on the score of rigor. The subject of convergency of series is one that seems to receive little attention in the colleges in this country, and the ignorance of students regarding it is often abysmal. The important question of uniform convergence is hardly hinted at in any English book, unless it be one of the recent treatises on theory of functions or Chrystal's algebra. This is certainly not as it should be, and it is to be hoped that this lack will soon be filled. Professor Byerly gives a proof of convergency in a special case which could without much trouble have been made general. He gives a page of cuts showing the gradual approximation to the representation of several functions, as terms are successively added to the series, which are very interesting.

The second part of the book is devoted to spherical harmonics, and to Bessel's and Lamé's functions. All the principal properties of spherical harmonics are given, while what is unnecessary is omitted. Here we again find an instructive plate giving graphical representations of the first seven Legendre's polynomials, or zonal harmonics, which will be appreciated by the student. Similar figures for Bessel's functions would have been instructive. There are also figures representing incomplete developments which are interesting by comparison with those previously mentioned.

In the chapter on Lamé's functions it is to be regretted that symmetry has not been preserved in relation to the three ellipsoidal co-ordinates, as has been most elegantly done in Halphen's "Traité des Fonctions Elliptiques."

The book is just what it purports to be. The preface states that the first part is based on Riemann-Hattendorff, and it includes besides a great deal not there treated. It is a clear, compact treatment of its subject-matter, and will be of great value to students of mathematical physics and to all persons who have to perform calculations of the kind considered. It contains those things that the "business" mathematical physicist wants to know, so arranged that he can find them at once. It is in addition much more interesting than such books have generally been. Heine's and Thomson and Tait's have been the standard treatises on spherical harmonics, but no one could pretend that Heine's was an attractive book to read, or Thomson and Tait's easy. Byerly's book is crowded with physical problems of all sorts, mostly worked out in detail. A good opportunity is also given the student to exercise himself in real numerical calculation by which he may get a tangible idea of the processes involved. A series of valuable tables of the values of the various functions is also given. Last, and not least in value, is to be mentioned the historical summary contributed by Dr. Maxime Bôcher, giving an admirable sketch of the whole subject, with a bibliography.

The book is well and clearly printed, and attractive in appearance (to one, as was stated at the beginning, who likes that sort of thing). Misprints are rare. On page 91 Ångström appears as Ängstrom, which spoils the pronunciation.

It may be mentioned that the historical essay on trigonometric series mentioned on page 61 is to be found in the *Bulletin des Sciences mathématiques* for 1880.

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NOTE ON SMITH'S REVIEW OF CAJORI.

BY PROF. GEORGE BRUCE HALSTED.

THE review, in the May BULLETIN, of Cajori's History of Mathematics by Professor David Eugene Smith produces an unfair impression. The facts upon which he says he bases his "harsh statement" do not justify it; and what he states as his "*facts*" are in large part not facts, but specimens of Professor Smith's *petitio principii.*

His ***First*** is, "The work is, in very considerable measure, merely a paraphrase of portions of *better* works." He then cites Ball and Fink, neither of which is a "better" work; in fact, apart from mere printer's slips in printing foreign words, Cajori is superior in point of actual accuracy to either of them, and incomparably superior in range.

Contrast Smith's *First* with his *Second:* "One has a right to expect a rich set of references to the standard literature of the day. Such references are offered by other histories, however humble, and every student needs them. Yet in this work there is not a single reference by volume and page." But neither is there in Fink, though just cited among those *better* than Cajori.

Such references were deliberately abjured by Cajori as interfering too seriously with the popular character and readableness of his book.

As for Smith's Dictionary of Biography, this is a slip of Professor Smith, see 9 under Books of Reference. Had Professor Cajori waited to obtain and study all the works of all the men whose names are so easy for Mr. Smith to mention, he might have died of old age, instead of giving us a charming history of mathematics.

That Mr. Smith himself is not above criticism I judge from the fact that he does not perceive the most serious error in Cajori's book, pointed out in the *Educational Review.*

Mr. Smith's *Third:* "But however charitable the reader may be, he will close the final chapters with even greater disappointment than he experienced in reading the earlier ones." I say, on the contrary, that these very chapters for successful condensed popular statement of the ground they cover are without rival in the world, in any language.

"While . . . thirty American mathematicians could not be found who would wish to be mentioned in a work which ignores the names of so many world-known promoters of the science;"—the word "ignores" for "omits" seems unfair, and 30 is perhaps a slip for 300.

His "*Fourth.* A final reason . . .: the work is carelessly written." This I beg leave to categorically deny. For example, Chasles' Christian name is given correctly. But it appears that "carelessly" is an exaggeration for "not with the *greatest* care."

"In the bibliography, . . . errors will be found in" . . . 65 [u for ü], 67, 84, 88, 97 [in all four the same small *t* omitted from the same word by the printer], 96 [B for R]. Thus the row of "numbers" is seen to be unfair.

In the words of the *Nation's* review: "But these are trifling faults. What we have a right to expect in such a handbook is an agreeable narrative of the most material events in the history of mathematics, and this Professor Cajori incontestably supplies."

When two reviewers hold such opposite opinions, and express their views in a manner so radically different, as Professor Halsted and the writer do concerning Cajori's History of Mathematics, the question becomes merely a matter for the reader to decide. It is submitted that Dr. Halsted's statements invalidate in no single point the review published in the May number.

As regards the first objection, the works of Ball and Fink are not the only, nor even the chief, ones referred to, as Dr. Halsted's article seems to imply. Indeed the quotation from Fink is rather to illustrate the faulty translation, that author not being mentioned among those from whom Professor Cajori has chiefly drawn. The works of Gow, Hankel, and Cantor, which will probably be admitted to be "better" works, are the ones to which, with Ball, the most prominent reference is made. But the real point of the criticism is not as to which of any two works is the better, but as to the way in which various works have been laid under contribution. If Professor Cajori's book contains such close paraphrases of inferior histories, so much the worse. While it is foreign to the real issue, the writer would not have it inferred from any silence of his that he for a moment subscribes to the proposition that "Cajori is superior in point of actual accuracy" to the second edition of Ball.

As to the second point, it is not clear why it should have interfered with "the popular character and readableness of his book" if such a bibliographical reference as that mentioned on p. 193 of the BULLETIN had been made more explicit, or if references had been given whenever an extract was made. Can any one justly affirm that the references as they stand are of any material value? Replying further to Professor Halsted, it is reasserted that the valuable *articles* (as distinguished from the valuable *article* mentioned) in Smith's Dictionary of Biography are worthy of mention. That the author should have "waited to obtain and study all the works of all the men" mentioned was not asserted. The criticism distinctly refers to the "sweeping omission"—to the fact that not one of them was consulted. But surely Professor Halsted, who has himself done so much for the history of mathematics in this country, will not contend but that every advanced scholar in this line is familiar with substantially every author named in the general list; and the professor must admit that one like himself could easily, with no hesitation, add several well-known names, and many of lesser note, to those mentioned.

The error pointed out in the article in the *Educational Review* can hardly, save on the principle of *ipse dixit*, be called "the most serious error in Cajori's book." It is, however, quite pardonable for Professor Halsted to be partial to that article; he wrote it. As to the article in the *Nation*, that was a charming essay, but it was hardly a serious review of Cajori.

Since with the exception of a few pages in Ball we have no similar attempt at a synopsis of the whole of modern mathematical history up to the present time, it is quite safe to make the sweeping assertion that this portion of the work is "without a rival in the world, in any language." Nevertheless one may close the final chapters with disappointment.

Having no idea of the meaning of Professor Halsted's reference to Chasles' Christian name, or of his statement that "30 is perhaps a slip for 300," the writer ventures to pass them by. He also ventures to reassert his appreciation of the work under discussion as a popular exposition of the historical advance of mathematical science, as set forth in the closing paragraph of his review.

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ON ORTHOGONAL SUBSTITUTIONS.

BY PROF. HENRY TABER.

IN 1846, in *Crelle's Journal*, Cayley gave his well-known determination of the general proper orthogonal substitution of n variables rationally in terms of the minimum number of parameters. Subsequently, in *Crelle*, vol. 50, and in the *Philosophical Transactions* for 1858, Cayley expressed these results in the notation of matrices.

In accordance with the theory of matrices,[*] two linear substitutions are regarded as susceptible of being added or subtracted. If $(\phi)_{rs}$ denotes that coefficient of the linear substitution ϕ which appears in the rth row and sth column of its square array or matrix, the sum or difference of the linear substitutions ϕ and ψ is defined as follows:

$$(\phi \pm \psi)_{rs} = (\phi)_{rs} \pm (\psi)_{rs}.$$

Addition and subtraction of linear substitutions are then subject to the laws which hold for these processes when we deal with the symbols of ordinary algebra.

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[*] See Cayley's "Memoir on the Theory of Matrices," *Phil. Trans.*, 1858.

Multiplication is taken as equivalent to the composition of substitutions ; and consequently multiplication is associative and distributive, but not in general commutative.* The reciprocal ϕ^{-1} of any linear substitution ϕ is the linear substitution which multiplied by or into ϕ gives the identical substitution.

In the notation of matrices, Cayley's expression for the general proper orthogonal substitution of n variables is

$$(1+\Upsilon)^{-1}(1-\Upsilon),$$

in which 1 denotes the identical substitution (or matrix unity) and Υ is an arbitrary alternate (or skew symmetric) linear substitution, but such that the determinant of $1+\Upsilon$ is not zero.

For if we denote by tr ϕ the substitution transverse (or conjugate) to ϕ, that is, the substitution whose matrix is obtained from the matrix of ϕ by interchanging its rows and columns, we then have

$$\begin{aligned}\operatorname{tr}\phi &= (1-\operatorname{tr}\Upsilon)(1+\operatorname{tr}\Upsilon)^{-1}\\ &= (1+\Upsilon)(1-\Upsilon)^{-1}\,\dagger\\ &= (1-\Upsilon)^{-1}(1+\Upsilon);\ddagger\end{aligned}$$

and therefore

$$\operatorname{tr}\phi = \phi^{-1},$$

that is

$$\phi\operatorname{tr}\phi = 1,$$

which is the condition that ϕ shall be orthogonal.

The orthogonal substitutions given by Cayley's expression are all proper. And every orthogonal substitution whose characteristic equation does not have -1 as a root is given by this expression; but an orthogonal substitution whose characteristic equation has -1 as a root cannot be put in this form.§

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* If $\phi\psi = \psi\phi$, and if $f\phi$ and $F\psi$ are polynomials in ϕ and ψ respectively, then $f\phi \,.\, F\psi = F\psi \,.\, f\phi$. In particular, ϕ is commutative with any polynomial in ϕ.

† For $\operatorname{tr}(\phi\psi) = \operatorname{tr}\psi \,.\, \operatorname{tr}\phi$, $\operatorname{tr}(\phi^{-1}) = (\operatorname{tr}\phi)^{-1}$, and $\operatorname{tr}(\phi+\psi) = \operatorname{tr}\phi + \operatorname{tr}\psi$; moreover, $\operatorname{tr} 1 = 1$, and $\operatorname{tr}\Upsilon = -\Upsilon$, by definition, since (employing the notation used above) $(\Upsilon)_{rs} = -(\Upsilon)_{sr}$.

‡ $(1-\Upsilon)^{-1}$ can be expressed as a polynomial in powers of Υ; it is therefore commutative with $1+\Upsilon$. See note above.

§ If -1 is not a root of the characteristic equation of the orthogonal substitution ϕ, $1+\phi$ has a reciprocal, and we may put

$$\Upsilon = (1-\phi)(1+\phi)^{-1}.$$

About a third of a century after Cayley's paper of 1846, the investigation was completed by Frobenius, who showed, *Crelle*, vol. 84, that every proper orthogonal substitution not given by Cayley's expression is given by the limit of Cayley's expression when Υ is infinite. That is, let ϕ be any proper orthogonal substitution whose characteristic equation has -1 as a root (in which case ϕ is not given by Cayley's expression), then we can always find a skew symmetric linear substitution Υ_ρ whose coefficients are rational functions of a parameter ρ, of which one at least is infinite for $\rho = 0$, such that $(1 + \Upsilon_\rho)^{-1}(1 - \Upsilon_\rho)$ can be made as nearly as we please equal to ϕ by taking ρ sufficiently small. Moreover, we have

$$\phi = L_{\rho=0}(1 + \Upsilon_\rho)^{-1}(1 - \Upsilon_\rho).$$

In what follows we shall be concerned principally with a consequence of this theorem.

By a well-known theorem every orthogonal substitution whatever is equal to the product of Cayley's expression into a symmetric orthogonal substitution. That is, if ϕ is orthogonal, we can always put

$$\phi = \phi_0(1 + \Upsilon)^{-1}(1 - \Upsilon),$$

in which Υ is skew symmetric, and $\phi_0 = \text{tr}\ \phi_0$, $\phi_0^2 = 1$ (whence follows $\phi_0\ \text{tr}\ \phi_0 = 1$). If the characteristic equation of ϕ has not -1 as a root, we may put $\phi_0 = 1$.

We may so choose ϕ_0 and Υ that they shall be commutative.* In which case

$$\phi^2 = \phi_0^2[(1 + \Upsilon)^{-1}(1 - \Upsilon)]^2 = [(1 + \Upsilon)^{-1}(1 - \Upsilon)]^2.$$

From which we obtain

$$\begin{aligned} \text{tr}\ \Upsilon &= (1 + \text{tr}\ \phi)^{-1}(1 - \text{tr}\ \phi) \\ &= (1 + \phi^{-1})^{-1}(1 - \phi^{-1}) \\ &= (\phi + 1)^{-1}\ \phi . \phi^{-1}(\phi - 1) \\ &= (\phi - 1)(\phi + 1)^{-1} \\ &= -\Upsilon; \end{aligned}$$

whence it follows that Υ is skew symmetric. We also have

$$(1 + \Upsilon)\ \phi = 1 - \Upsilon.$$

But since

$$1 + \Upsilon = [(1 + \phi) + (1 - \phi)](1 + \phi)^{-1} = 2(1 + \phi)^{-1},$$

the determinant of $1 + \Upsilon$ is not zero. Therefore

$$\phi = (1 + \Upsilon)^{-1}(1 - \Upsilon).$$

[*] If -1 is a root of multiplicity m of the characteristic equation of ϕ,

Whence it follows that every orthogonal substitution which is the second power or square of an orthogonal substitution is given by the square of Cayley's expression. Thus the square of every improper orthogonal substitution is also the square of a proper orthogonal substitution.

But every proper orthogonal substitution cannot be put equal to the square of Cayley's expression. *Exempli gratia*, the proper orthogonal substitution given on p. 123 of vol. 16 of the *American Journal of Mathematics* is not the second power of any orthogonal substitution whatever.

We are thus led to designate an orthogonal substitution as of the first or second kind according as it is or is not the square of an orthogonal substitution.

In a paper read at last year's Mathematical Congress at Chicago I have shown that every real proper orthogonal substitution of any number of variables, and every imaginary proper orthogonal substitution of two or three variables, also every proper symmetric orthogonal substitution, is given by the square of Cayley's expression; and in a recent number of the *American Journal of Mathematics*, that every orthogonal substitution given by Cayley's expression is also given by the square of Cayley's expression.* Each of the proper orthogonal substitutions enumerated above is therefore the second power of a (proper) orthogonal substitution.

If ϕ is any proper orthogonal substitution of the second kind, by the theorem of Frobenius, a skew symmetric linear substitution Υ_ρ can be found whose coefficients are rational functions of a parameter ρ such that

$$\phi_\rho = (1 + \Upsilon_\rho)^{-1}(1 - \Upsilon_\rho)$$

can be made as nearly as we please equal to ϕ by taking ρ sufficiently small. And since ϕ_ρ is given by Cayley's expression, we can find a proper orthogonal substitution ψ_ρ whose coefficients are algebraic functions of ρ such that

$$\phi_\rho = \psi_\rho \psi_\rho. \dagger$$

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and if the roots of this equation other than -1 are $g_1, g_2, \ldots g_t$, then we may put

$$\phi_0 = 1 - 2\Phi,$$

in which

$$\Phi = \frac{(\phi+1)^m - (g_1+1)^m}{-(g_1+1)^m} \cdot \frac{(\phi+1)^m - (g_2+1)^m}{-(g_2+1)^m} \cdots \frac{(\phi+1)^m - (g^t+1)^m}{-(g_t+1)^m}.$$

* For if ϕ is an orthogonal substitution given by Cayley's expression, it is the square of an orthogonal substitution. But then by the preceding theorem ϕ is given by the square of Cayley's expression. The coefficients of the orthogonal substitution given by Cayley's expression of which ϕ is the second power are algebraic functions of the coefficients of ϕ.

† See preceding note.

Therefore, by taking ρ sufficiently small we can make the second power of ψ_ρ as nearly equal to ϕ as we please. Whence it follows that an orthogonal substitution of the first kind can always be found whose second power shall be as nearly as we please equal to any proper orthogonal substitution of the second kind (itself not the square of any orthogonal substitution).

As an example, let ϕ be the proper orthogonal substitution whose matrix is

$$\begin{array}{cccc}
\dfrac{1}{2c^2}-1, & \dfrac{\sqrt{-1}}{2c^2}, & 0, & \dfrac{1}{c} \\[2ex]
\dfrac{\sqrt{-1}}{2c^2}, & -\dfrac{1}{2c^2}-1, & 0, & \dfrac{\sqrt{-1}}{c} \\[2ex]
0, & 0, & -1, & 0 \\[2ex]
-\dfrac{1}{c}, & \dfrac{-\sqrt{-1}}{c}, & 0, & -1
\end{array}$$

in which c is an arbitrary constant. This orthogonal substitution is not the square of any orthogonal substitution whatever. But if ψ_ρ designates the orthogonal substitution whose matrix is

$$\begin{array}{cccc}
\dfrac{\rho}{r}+\dfrac{1}{8\rho r^3c^2}, & \dfrac{1}{r}+\dfrac{i}{8\rho r^3c^2}, & \dfrac{-i}{4\rho rc}+\dfrac{1-\rho i}{4r^3c}, & \dfrac{1}{4\rho rc}-\dfrac{i+\rho}{4r^3c} \\[2ex]
\dfrac{-1}{r}+\dfrac{i}{8\rho r^3c^2}, & \dfrac{\rho}{r}-\dfrac{1}{8\rho r^3c^2}, & \dfrac{1}{4\rho rc}+\dfrac{i+\rho}{4r^3c}, & \dfrac{i}{4\rho rc}+\dfrac{1-\rho i}{4r^3c} \\[2ex]
\dfrac{i}{4\rho rc}+\dfrac{1+\rho i}{4r^3c}, & \dfrac{-1}{4\rho rc}+\dfrac{i-\rho}{4r^3c}, & \dfrac{\rho}{r}, & \dfrac{1}{r} \\[2ex]
\dfrac{-1}{4\rho rc}-\dfrac{i-\rho}{4r^3c}, & \dfrac{-i}{4\rho rc}+\dfrac{1+\rho i}{4r^3c}, & \dfrac{-1}{r}, & \dfrac{\rho}{r}
\end{array}$$

in which $r=\sqrt{1+\rho^2}$ and $i=\sqrt{-1}$, then the matrix of $\phi_\rho=\psi_\rho^2$ is

$$\begin{array}{cccc}
\dfrac{\rho^4-1}{r^4}+\dfrac{1}{2c^2r^4}, & \dfrac{2\rho(\rho^2+1)}{r^4}+\dfrac{i}{2c^2r^4}, & \dfrac{-\rho i(\rho+i)}{cr^4}, & \dfrac{1-\rho i}{cr^4} \\[2ex]
\dfrac{-2\rho(\rho^2+1)}{r^4}+\dfrac{i}{2cr^4}, & \dfrac{\rho^4-1}{r^4}-\dfrac{1}{2c^2r^4}, & \dfrac{\rho(\rho+i)}{cr^4}, & \dfrac{i(1-\rho i)}{cr^4} \\[2ex]
\dfrac{\rho(1+\rho i)}{cr^4}, & \dfrac{\rho i(1+\rho i)}{cr^4}, & \dfrac{\rho^4-1}{r^4}, & \dfrac{2\rho(1+\rho^2)}{r^4} \\[2ex]
\dfrac{-1+\rho i}{cr^4}, & \dfrac{-i(1+\rho i)}{cr^4}, & \dfrac{-2\rho(1+\rho^2)}{r^4}, & \dfrac{\rho^4-1}{r^4}
\end{array}$$

and if ρ is taken sufficiently small $\phi_\rho = \psi_\rho^2$ may be made as nearly as we please equal to ϕ.

So long as ρ is not zero, the orthogonal substitution ϕ_ρ is of the first kind, that is, ϕ_ρ is the second power of the orthogonal substitution ψ_ρ. But in the limit for $\rho = 0$, $\phi_\rho = \phi$ is of the second kind, that is, ϕ_ρ, for $\rho = 0$, is not the square of any orthogonal substitution.

The orthogonal substitutions of the first kind have certain interesting properties. Any orthogonal substitution of this kind has an orthogonal *m*th root (*m* being any positive integer)—meaning by the *m*th root of a linear substitution ϕ the linear substitution ψ satisfying the equation $\phi = \psi^m$.

This theorem may be proved as follows. Let e^θ denote the infinite series $\overset{\infty}{\underset{0}{\Sigma}} r \frac{\theta^r}{r!}$, convergent for any linear substitution. For any linear substitution ϕ whose determinant does not vanish, a polynomial $f(\phi)$ in integer powers of ϕ can be found such that $\phi = e^{f(\phi)}$. If now ϕ is any orthogonal substitution of the first kind, we have

$$\phi = [(1 + \Upsilon)^{-1}(1 - \Upsilon)]^2,$$

in which Υ is skew symmetric and such that the determinant of $1 - \Upsilon$ does not vanish. Therefore we may find a polynomial $\vartheta = f(\Upsilon)$ in powers of Υ such that

$$1 - \Upsilon = e^{\vartheta}.$$

Taking the transverse of either side we have

$$1 + \Upsilon = 1 - \operatorname{tr} \Upsilon = e^{\operatorname{tr} \vartheta}.$$

Whence it follows that

$$(1 + \Upsilon)^{-1}(1 - \Upsilon) = (e^{\operatorname{tr} \vartheta})^{-1} e^{\vartheta} = e^{-\operatorname{tr} \vartheta} e^{\vartheta} = e^{-\operatorname{tr} \vartheta + \vartheta},$$

since ϑ and $\operatorname{tr} \vartheta = f(\operatorname{tr} \Upsilon) = f(-\Upsilon)$* are both polynomials in Υ, and therefore are commutative.

Let $-f(-\Upsilon) + f(\Upsilon) = -\operatorname{tr} \vartheta + \vartheta = \frac{1}{2}\theta$; we then have

$$\phi = (e^{\frac{1}{2}\theta})^2 = e^{\theta},$$

in which θ is skew symmetric. If now

$$\psi = e^{\frac{1}{m}\theta},$$

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* See note, p. 253.

in which m is any positive integer, then, since $\frac{1}{m}\theta$ is also skew symmetric,

$$\begin{aligned}\psi \operatorname{tr} \psi &= e^{\frac{1}{m}\theta} \operatorname{tr} e^{\frac{1}{m}\theta} \\ &= e^{\frac{1}{m}\theta} e^{\frac{1}{m}\operatorname{tr}\theta} \\ &= e^{\frac{1}{m}\theta} e^{-\frac{1}{m}\theta} \\ &= e^{\frac{1}{m}\theta - \frac{1}{m}\theta} \\ &= 1.\end{aligned}$$

Moreover,

$$\phi = e^{\theta} = \left(e^{\frac{1}{m}\theta}\right)^m = \psi^m.$$

Therefore ϕ has an mth root which is orthogonal.

By taking m sufficiently great, the coefficients of $\frac{1}{m}\theta$ may be made as small as we please;* and therefore $\psi = e^{\frac{1}{m}\theta}$ may be made to approach as near as we please to the identical substitution. But we still have $\phi = \psi^m$. Whence it follows that every orthogonal substitution of the first kind can be generated by the repetition of the same infinitesimal orthogonal substitution.

An infinitesimal orthogonal substitution cannot have -1 as a root of its characteristic equation. The infinitesimal orthogonal substitutions are therefore of the first kind; and can all be put equal to e^{θ}, in which θ is infinitesimal and skew symmetric. But any power of an orthogonal substitution of the first kind is of that kind. Therefore the substitutions generated by the repetition of the same infinitesimal orthogonal substitution are of the first kind. Consequently orthogonal substitutions of the second kind cannot be generated in this way.

The orthogonal substitutions of the second kind cannot have a root of even index. But every orthogonal substitution of the second kind has a root with any odd index. For by the theorem given on p. 253 any orthogonal substitution ϕ of the second kind can be put equal to

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* Employing the notation of p. 251 we have $\left(\frac{1}{m}\theta\right)_{rs} = \frac{1}{m}\left(\theta\right)_{rs}$.

$$\phi_0(1+\Upsilon)^{-1}(1-\Upsilon),$$

in which ϕ_0 is symmetric and orthogonal, and Υ is skew symmetric; and moreover ϕ_0 and Υ are commutative. If now ϑ is a polynomial in Υ satisfying the equation

$$1+\Upsilon=e^{\vartheta},$$

then $\theta = \vartheta - \mathrm{tr}\ \vartheta$ is skew symmetric and

$$(1+\Upsilon)^{-1}(1-\Upsilon)=e^{\theta}.$$

But then since θ is a polynomial in Υ, it is commutative with ϕ_0. Consequently, if

$$\psi = \phi_0 e^{\frac{1}{2m+1}\theta}$$

(m being any positive integer),

$$\begin{aligned} \psi\,.\,\mathrm{tr}\ \psi &= \phi_0 e^{\frac{1}{2m+1}\theta} e^{-\frac{1}{2m+1}\theta}\phi_0 \\ &= e^{\frac{1}{2m+1}\theta} e^{-\frac{1}{2m+1}\theta} \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \phi &= \phi_0(1+\Upsilon)^{-1}(1-\Upsilon) \\ &= \phi_0 e^{\theta} \\ &= \left(\phi_0 e^{\frac{1}{2m+1}\theta}\right)^{2m+1} \\ &= \psi^{2m+1}. \end{aligned}$$

These results may be summarized as follows. Every proper orthogonal substitution is of the first or second kind according as it is or is not the second power of an orthogonal substitution. All proper orthogonal substitutions of two or three variables, all real proper orthogonal substitutions of any number of variables, all orthogonal substitutions given by Cayley's expression (including all infinitesimal orthogonal substitutions), and all proper symmetric orthogonal substitutions are of the first kind. Every orthogonal substitution of the first kind is given by the square of Cayley's expression, can be generated by the repetition of the same infinitesimal orthogonal substitution, and has an orthogonal mth root for any index m. No orthogonal substitution of the second kind can be generated by the repetition of the same infinitesimal orthogonal substitution. Every orthogonal substitution of the second kind has an orthogonal root with any odd index; and no orthogonal substitution of this kind has an orthogonal root with even index. But, corresponding to any proper orthogonal substitution ϕ of the second kind, can always be found an orthogonal substitution ψ_ρ of the first kind whose coefficients are algebraic functions of a parameter ρ such that, by taking ρ sufficiently small, the $2m$th power of ψ_ρ can be made as nearly as we please equal to ϕ. Moreover, we have

$$\phi = L_{\rho=0}\psi_\rho^{2m},$$

but not

$$\phi = \left[L_{\rho=0}\psi_\rho\right]^{2m}.$$

[An exactly similar theory holds for the linear substitutions which automorphically transform a bilinear form with cogrediant variables.]

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, June 2, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Mr. William Eimbeck, U. S. Coast and Geodetic Survey, Washington, D. C.; Professor Herman J. Gaertner, Indiana Normal College, Covington, Indiana; Mr. Henry Volkman Gummere, Swarthmore College, Swarthmore, Pa.; Mr. George Herbert Ling, Columbia College, New York. The by-laws were amended in accordance with the recommendations of the council, the amendments to go into effect July 1, 1894.

Dr. Henry Taber read a paper entitled "On orthogonal substitutions." This paper appears in the present number of the BULLETIN, see p. 251.

THE council of the Society, influenced by the high importance of most of the papers presented to the Mathematical Congress at Chicago in 1893, by the desirability of their publication collectively, prepared, as they were, to a large extent, for the purpose of giving a general survey of the present state of knowledge throughout almost the entire range of mathematics, and by a sense of the honor conferred upon America by the contributions of so many distinguished mathematicians resident abroad, has resolved to undertake the publication of a volume containing the hitherto unpublished papers read at the Congress.

Macmillan & Co. have been consulted and have offered to co-operate with the Society on very generous terms. The funds immediately available for the purpose, however, are insufficient, and it will be necessary to raise the additional sum required by private subscriptions. The publication committee of the Society, or that of the Chicago Congress, will be glad to furnish detailed information upon application.

THE following graduate courses in mathematics will be given at Johns Hopkins University during the academic year 1894–95 :—Professor Craig : (1) Theory of functions (advanced course), three times weekly, first half-year; (2) Abelian functions, three times weekly, first half-year ; (3) Differential equations, three times weekly, second half-year ; (4) Elliptic functions, three times weekly, second half-year ; (5) Mathematical seminary, two hours weekly, through the year. Professor Franklin: (1) Theory of functions (elementary course), twice weekly, through the year ; (2) Theory of invariants, three times weekly, through the year. Dr. Chessin: (1) Theoretical dynamics, three times weekly, second half-year ; (2) Partial differential equations, three times weekly, first half-year. Dr. Hulburt: (1) Theory of plane curves, four times weekly, first half-year ; (2) Surfaces and twisted curves (metrical), twice weekly, second half-year.

The following graduate courses in astronomy will be given: Dr. Poor : (1) The theory and use of astronomical instruments, three times weekly, first half-year ; (2) Advanced theoretical astronomy, three times weekly, second half-year ; (3) General course in theoretical and practical astronomy, twice weekly, through the year ; (4) Astronomical seminary, weekly, through the year ; (5) Practical work with the instruments, daily, through the year. Dr. Chessin : (1) Celestial mechanics, three times weekly, through the year. T. S. F.

AT the University of Chicago during the three quarters (a, w, s) of the academic year 1894–95, the following courses in advanced mathematics will be offered : by Professor Moore, Elliptic modular functions (a), Projective geometry (a), Configurations (s), Groups (s); by Professor Bolza, Theory of functions of a complex variable (a), Quaternions (a), Substitutions (w), Analytic geometry of three dimensions (w) ; by Assistant Professor Maschke, Higher plane curves (a), Advanced integral calculus (a-w), Weierstrass's theory of elliptic functions (w), Algebraic surfaces (s), Analytical mechanics (s); by Dr. Young, Theory of numbers (w), Theory of equations (w-s) ; by Dr. Boyd, Differential equations (s) ; by Mr. Hancock, Calculus of variations (w). E. H. M.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Appell (P.) et Goursat (E.). Théorie des fonctions algébriques et de leurs intégrales. Études des fonctions analytiques sur une surface de Riemann. Paris, Gauthier-Villars, 1894. 8vo. 10 and 256 pp. (1er fasc.) Illustrated. Fr. 14.00

Banning (R.). Ueber Kugel- und Cylinderfunktionen und deren Kettenbruchentwickelung. [Diss.] Bonn, 1894. 8vo. 33 pp.

Boncompagni (B.). Catalogo dei lavori di Enrico Narducci. Roma, Tipografia delle scienze matematiche e fisiche, 1893. 4to. 2 and 18 pp.

Brunel (G.). Note sur le nombre de points doubles que peut présenter le périmètre d'un polygone. Bordeaux, Gounouilhou, 1894. 8vo. 4 pp.

Burali-Forti (C.). Sulle classi ordinate e i numeri transfiniti. (*Rendiconti del Circolo matematico di Palermo*, vol. VIII, pp. 169–179.) 8vo.

Dittmar (P.). Der Ort der Brennpunkte eines Büschels von Kegelschnitten, das von einem Ebenenbüschel aus einem Kegel II. Ordnung ausgeschnitten wird. [Progr.] Oppenheim, 1894. 4to. 26 pp.

Garcia de Galdeano (Z.). La enseñanza de la ciencia matematica en la Universidad. Madrid, 1894. 8vo. 36 pp. Mk. 1.50

Grünfeld (E.). Ueber adjungierte Systeme linearer Differentialgleichungen mit einer unabhängig veränderlichen Grösse. [Progr.] Nikolsburg, 1893. 8vo. 20 pp.

Hagen (J. G.). Synopsis der höheren Mathematik. (In 4 Bänden.) Bd. II: Geometrie der algebraischen Gebilde. Berlin, 1894. 4to. 416 pp. Mk. 30.00

Holtze (A.). Kleine mathematische Abhandlungen. [Progr.] Naumburg, 1894. 4to. 18 pp. 1 plate.

Jahrbuch über die Fortschritte der Mathematik. Herausgegeben von E. Lampe. 23ster Bd. (Jahrgang 1891.) 2tes Heft. Berlin, G. Reimer, 1894. 8vo. pp. 529–896. Mk. 9.00

Jordan (C.). Cours d'analyse de l'École polytechnique. 2e édition, entièrement refondue. Vol. 2: Calcul intégral. Paris, Gauthier-Villars, 1894. 8vo. 18 and 627 pp. Illustrated. Fr. 17.00

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Láska (W.). Einführung in die Funktionentheorie. Eine Ergänzung zu allen Lehrbüchern der Differential- u. Integralrechnung. Stuttgart, Maier, 1894. 8vo. 55 pp. Illustrated. Mk. 1.50

Papelier (G.). Leçons sur les coordonnées tangentielles. Avec une préface de M. P. Appell. Première partie: Géométrie plane. Paris, Nony, 1894. 8vo. 6 and 329 pp.

Schilling (F.). Beiträge zur geometrischen Theorie der Schwarz'schen s-Funktion. [Diss.] Göttingen, 1894. 8vo. 100 pp. Illustrated. Mk. 8.00

Schuster (P.). Sammlung von geometrischen Oertern als Uebungsbeispiele für die analytische Geometrie der Ebene. Glogau, 1894. 8vo. 22 pp. Mk. 1.20

Seeger (H.). Bemerkungen über Abgrenzung und Verwertung des Unterrichts in den Elementen der Infinitesimalrechnung. Güstrow, Opitz, 1894. 8vo. 24 pp. Illustrated. Mk. 0.50

Staeger (P.). Ueber die durch die Gleichung $y = \frac{\lambda}{3r^3}(r^2 - x^2)^{\frac{3}{2}}$ dargestellte Kurve. [Progr.] Eisleben, 1894. 4to. 21 pp.

Steinitz (E.). Ueber die Construction der Configurationen n_3. [Diss.] Breslau, 1894. 8vo. 45 pp.

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Exercices de géométrie à l'usage des élèves des classes de lettres. Pau, Broise [1894]. 12mo. 92 pp.

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Lenfant (E.). Cours d'algèbre élémentaire conforme au programme de l'École de Versailles, à l'usage des candidats aux écoles militaires du gouvernement et aux baccalauréats. Paris, Baudoin [1893]. 18mo. 189 pp. Fr. 3.00

——— Cours de géométrie descriptive conforme au programme de l'École de Versailles, suivi de compléments, à l'usage des candidats aux baccalauréats et aux écoles du gouvernement. Paris, Baudoin, 1894. 18mo. 164 pp. Illustrated. Fr. 3.50

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Roeder (H.). Trigonometrische und stereometrische Lehraufgabe der Unter-Sekunda. Zum Gebrauche an höheren Lehranstalten nach den neuen preussischen Lehrplänen bearbeitet. Breslau, Hirt, 1894. 8vo. 40 pp. Illustrated. Mk. 0.50.

Särchinger (E.). Beitrag zur Theorie der Funktionen des elliptischen Cylinders. [Progr.] Chemnitz, 1894. 4to. 28 pp.

Schlömilch (O.). Fünfstellige logarithmische und trigonometrische Tafeln. Wohlfeile Schulausgabe. 12te Auflage. Braunschweig, Vieweg, 1894. 8vo. 151 pp. Mk. 1.00

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Ewing (J. A.). The steam engine and other heat engines. Cambridge, Univ. Press (New York, Macmillan), 1894. 8vo. 400 pp. $3.75

Galilei (G.). Le opere. Edizione nazionale sotto gli auspizi di sua Maestà il Re d'Italia. Vol. IV. Firenze, Barbèra, 1894. 4to. 794 pp. Illustrated.

Gänge (C.). Die Polarisation des Lichtes. Kurze Darstellung ihrer Lehre und Anwendungen. Leipzig, Quandt & Händel. 8vo. 78 pp. Illustrated. Mk. 1.80

Goodwin (H. B.). The "ex-meridian" treated as a problem in dynamics. London, Philip, 1894. 8vo. 1s 6d.

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Klingatsch (A.). Die graphische Ausgleichung bei der trigonometrischen Punktbestimmung durch Einschneiden. Wien, Gerold, 1894. 8vo. 47 pp. 4 plates. Illustrated. Boards. Mk. 3.00

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Theil II: Elektrodynamik. Abschnitt 1: Theorie des permanenten Magnetismus und der constanten elektrischen Ströme. Berlin, 1894. 8vo. 124 pp. Mk. 3.00

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Kummell (C. H.). On the direct synthetical method of adjusting a triangulation. (U. S. Coast and Geodetic Survey, Appendix 12 to the Report for 1892.) Washington, Government Printing Office, 1893. 8vo. pp. 534–552.

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Mannheim (A.). Principes et développements de géométrie cinématique, ouvrage contenant de nombreuses applications à la théorie des surfaces. Paris, Gauthier-Villars, 1894. 4to. 9 and 591 pp. Illustrated. Fr. 25.00

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Rankine (W. J. M.). A manual of civil engineering. 19th edition, thoroughly revised by W. J. Millar. London, Griffin, 1894. 8vo. 836 pp. 16s.

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Simon (H. T.). Ueber Dispersion ultravioletter Strahlen. [Diss.] Berlin, 1894. 8vo. 51 pp. 3 plates.

Specific subject tests: Mechanics, stage I, containing a series of graduated tests on the various divisions of the subject, together with a number of typical government tests. London, Simpkin, 1894. 12mo. 16 pp. 1d.

Strehl (K.). Theorie des Fernrohrs auf Grund der Beugung des Lichts. 1ster Theil. Leipzig, Barth, 1894. 8vo. 8 and 136 pp. 1 plate. Mk. 4.00

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THIRD ANNUAL LIST OF PUBLISHED PAPERS

READ BEFORE THE NEW YORK MATHEMATICAL SOCIETY, TOGETHER WITH THE PLACES OF THEIR PUBLICATION.

Craig, J. E. Negative-Reciprocal Equations. Read June 3, 1893. *Proceedings of the United States Naval Institute*, vol. 19, No. 2, pp. 201–212, 1893.

Craig, Thomas. Note on Monogenic Functions of a Single Variable. Read Oct. 7, 1893. *Bulletin of the New York Mathematical Society*, vol. 3, No. 3, pp. 78–79, Dec., 1893.

Durell, Fletcher. Application of the New Education to the Differential and Integral Calculus. Read Nov. 4, 1893. *The American Mathematical Monthly*, vol. 1, No. 1, pp. 15–19, Jan., No. 2, pp. 37–41, Feb., 1894.

Echols, W. H. Note on the Theory of Functions. Read Oct. 7, 1893. *Annals of Mathematics*, vol. 8, No. 2, pp. 45–51, Jan., 1894.

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Safford, T. H. Instruction in Mathematics, as affected by Modern Algebraic and Geometrical Developments and by the Progress of the Physical Sciences. Read April 1, 1893. *Bulletin of the New York Mathematical Society*, vol. 3, No. 1, pp. 4–8, Oct., 1893.

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